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Improved Two-Point Block Backward Differentiation Formulae for Solving First Order Stiff Initial Value Problems of Ordinary Differential Equations

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Abstract
Two numerical methods- I2BBDF2 and I22BBDF2 that compute two points simultaneously at every step of integration by first providing a starting value via fourth order Runge-Kutta method are derived using Taylor series expansion. The two-point block schemes are derived by modifying the existing I2BBDF (5) method of Mohamad *et al.,* (2018). Convergence and stability analysis of the new methods are established with the methods being of order two and A-stable in both cases. Despite the very low order of the new methods, the accuracy of these methods on some stiff initial value problems in the literature proves their superiority over existing methods of higher orders such as I2BBDF(5), BBDF(5), E2OSB(4) among others.

Key words: A-stability, convergence, implicit block method, stiff initial value problems, order.

Introduction

Let us consider a system of first order stiff initial value problems of the form

$$y' = f(x, y),\ x \in [a, b],\ y(x_0) = y_0 \qquad 1$$

where $f : [x_n, x_N] \times \Re^m \to \Re^m$ is continuous and differentiable. However, f is assumed to satisfy the existence and uniqueness theorem within the interval of [,]ab. The system (1) can be regarded as stiff if its exact solution contains very fast and as well very slow components (Dahlquist, 1974). Solutions for stiff IVPs are usually characterized by the presence of transient and steady state components, which restrict the step-size of many numerical methods except methods with A-stability properties (Musa, *et al.,* 2013a, Suleiman *et al.,* 2015). This behaviour makes it difficult to develop suitable methods for solving stiff initial value problems of ordinary differential equations (ODEs). However, efforts have been made by such researchers as Adesanya *et al.,* (2012), Ibrahim *et al.,* (2007), Musa *et al.,* (2012), Musa, *et al.,* (2013b), Nazir *et al.,* (2012), Suleiman *et al.,* (2015), Zawawi *et al.,* (2012), among others to develop numerical methods for solving stiff problems. The need to obtain numerical results in terms of maximum error have also attracted the attention of such researchers as Babangida *et al.,* (2016), Ibrahim *et al.,* (2007), Majid and Muktar, (2017), Musa *et al.,* (2013a) and Mohamad *et al.,* (2018) among others.

The motivation of this research is to modify the method developed by Mohamad *et al.,* (2018) so as to improve its accuracy in terms of maximum error by using a different strategy, which is to reduce the back values introduced from four to two, provide starting values for the modified methods using fourth order Runge-Kutta method and establish stability properties, among others to compare with some existing methods. Section 2 contains the derivation of the method while section 3 discusses the stability analysis and convergence of the methods. The implementation of the methods is presented in section 4 while section 5 deals with the test problems which are some stiff problems and the numerical results in comparison to some existing methods. Sections 6 and 7 deal with the discussion and conclusion respectively.

Derivation of the Improved Two-Point Methods

The derivation of the two-point block methods is done by considering the method developed by Mohamad *et al.,* (2018) whereby we reduce the four back values from $y_{n-3}\, y_{n-2}\, y_{n-1}$, and y_n to two back values y_{n-1} and y_n for solving (1). This is to improve the accuracy of the method in terms of error and reduce the computational burden.

To construct the two-point block methods, Lambert (1973) defined linear multistep method (LMM) of step number k as

$$\sum_{j=0}^{k} \underline{a}_j y_{n+j} = h \sum_{j=0}^{k} \underline{b}_j f_{n+j} \qquad 2$$

where, α_j and β_j are constants with $\alpha_k \neq 0$ and that not both α_0 and β_0 are zero.

The two-point block methods are derived by representing (2) in the form of block multistep method, with two cases considered:

$$\sum_{j=0}^{k} \underline{a}_{j,i} y_{n+j-1} = h \underline{b}_l (f_{n+l} - \underline{r}\, f_{n+l-1}) \qquad 3$$

where α_{ji}, β_j are coefficients of and y_n and f_n respectively. In (3), ρ is a free parameter that will be chosen in the interval (1,1)–as stated by Vijitha-Kumara (1985).

We present below the two-point formula of the existing method developed by Mohamad *et al.,* (2018)

$$\left.\begin{aligned} y_{n+1} &= -\frac{1}{73} y_{n-3} + \frac{11}{146} y_{n-2} - \frac{6}{73} y_{n-1} + \frac{82}{73} y_n - \frac{15}{146} y_{n+2} + \frac{42}{73} hf_n + \frac{48}{73} hf_{n+1} \\ y_{n+2} &= \frac{15}{236} y_{n-3} - \frac{23}{59} y_{n-2} + y_{n-1} - \frac{78}{59} y_n + \frac{389}{236} y_{n+1} + \frac{21}{59} hf_{n+1} + \frac{24}{59} hf_{n+2} \end{aligned}\right\} \qquad 4$$

Thus, we present below the two cases of our method formulation.

Case 1: In (3), set k = 2,i = l = 1 which gives the following linear difference operator, L as:

$$\left.\begin{aligned} L[y(x_n, h)] &= \sum_{j=0}^{2} \alpha_{j,1} y_{n+j-1} - h\beta_1 (f_{n+1} - \rho f_n) \\ &= \alpha_{0,1} y_{n-1} + \alpha_{1,1} y_n + \alpha_{2,1} y_{n+1} - h\beta_1 (f_{n+1} - \rho f_n) \\ &= \alpha_{0,1} y(x_n - h) + \alpha_{1,1} y(x_n) + \alpha_{2,1} y(x_n + h) - h\beta_1 (f(x_n + h) - \rho f(x_n)) \end{aligned}\right\} \qquad 5$$

where $\alpha_{2,1} = 1$.

Expanding $y(x_n - h), y(x_n + h), y(x_n), f(x_n + h)$ and $f(x_n)$ in (5) using Taylor series and collect like terms in $y(x_n), y'(x_n), y''(x_n), y'''(x_n), \ldots,$ yields the following linear operator:

$$L_1[y(x_n), h] = C_{0,1}y(x_n) + C_{1,1}hy'(x_n) + C_{2,1}y''(x_n) + \ldots = 0 \qquad 6$$

where,

$$\left.\begin{aligned} C_{0,1} &= a_{0,1} + a_{1,1} = -1 \\ C_{1,1} &= \frac{1}{2}a_{0,1} - b_1 = -\frac{1}{2} \\ C_{2,1} &= b_1(r-1) - a_{0,1} = -1, \text{ since } a_{2,1} = 1 \end{aligned}\right\} \qquad 7$$

Solving (7) simultaneously and choosing $\rho = -\frac{1}{5}$ in a Maple software environment gives

$$\left.\begin{aligned} \alpha_{0,1} &= \frac{1}{4} \\ \alpha_{1,1} &= -\frac{5}{4} \\ \beta_1 &= \frac{5}{8} \end{aligned}\right\} \qquad (8)$$

Case 2: In (3), set $k = 3, i = l = 2$ and in a s imilar manner, we obtain the coefficient for second point as follows taking $\alpha_{3,2} = 1$:

$$\left.\begin{aligned} \alpha_{0,2} &= -\frac{3}{19} \\ \alpha_{1,2} &= \frac{13}{19} \\ \alpha_{2,2} &= -\frac{29}{19} \\ \beta_2 &= \frac{10}{19} \end{aligned}\right\} \qquad (9)$$

Substituting (8) and (9) into (5) gives the new improved two -point block backward differentiation formula (I2BBDF) formulated thus:

$$\left.\begin{aligned} y_{n+1} &= -\frac{1}{4}y_{n-1} + \frac{5}{4}y_n + \frac{1}{8}hf_n + \frac{5}{8}hf_{n+1} \\ y_{n+2} &= \frac{3}{19}y_{n-1} - \frac{13}{19}y_n + \frac{29}{19}y_{n+1} + \frac{2}{19}hf_{n+1} + \frac{10}{19}hf_{n+2} \end{aligned}\right\} \qquad (10)$$

Also, choosing $\rho = -\frac{1}{6}$ in cases 1 and 2 gives another new improved two-point block backward differentiation formula (I22BBDF) formulated as

$$\left.\begin{aligned} y_{n+1} &= -\frac{5}{19}y_{n-1} + \frac{24}{19}y_n + \frac{2}{19}hf_n + \frac{12}{19}hf_{n+1} \\ y_{n+2} &= \frac{11}{68}y_{n-1} - \frac{12}{17}y_n + \frac{105}{68}y_{n+1} + \frac{3}{34}hf_{n+1} + \frac{9}{17}hf_{n+2} \end{aligned}\right\} \qquad (11)$$

Stability Analysis and Convergence of the New Methods

We present the stability analysis of methods (10) and (11) where zero stability and A - stability is defined.

Definition 3.1 *A linear multistep method (LMM) is said to be zero stable if no root of the first characteristics polynomial has modulus greater than one and that any root with modulus one is simple* (Babangida *et al.,* 2016).

Definition 3.2 *A linear multistep method (LMM) is said to be A-stable if its stability region covers the entire negative half-plane* (Babangida *et al.,* 2016).

The I2BBDF2 formula (10) and I22BBDF2 formula (11) can each be written in matrix form

$$A_0 Y_m = A_1 Y_{m-1} + h(B_0 F_{m-1} + B_1 F_m) \tag{12}$$

where, for (10),

$$A_0 = \begin{pmatrix} 1\,0 & -\frac{29}{19} & 1 \end{pmatrix};\ A_1 = \begin{pmatrix} -\frac{15}{44} & - \\ \frac{313}{1919} & -— \end{pmatrix};\ B_0 = \begin{pmatrix} 0 & \frac{1}{8} \\ 00 & \end{pmatrix};\ B_1 = \begin{pmatrix} \frac{5}{8} & 0 \\ \frac{210}{1919} & — \end{pmatrix}$$

while, for (11),

$$A_0 = \begin{pmatrix} 1 & 0 \\ -\frac{105}{68} & 1 \end{pmatrix};\ A_1 = \begin{pmatrix} -\frac{5}{19} & \frac{24}{19} \\ \frac{11}{68} & -\frac{12}{17} \end{pmatrix};\ B_0 = \begin{pmatrix} 0 & \frac{2}{19} \\ 00 & \end{pmatrix};\ B_1 = \begin{pmatrix} \frac{12}{19} & 0 \\ \frac{3}{34} & \frac{9}{17} \end{pmatrix}$$

and $Y_m = \begin{pmatrix} y_{n+1} \\ y_{n+2} \end{pmatrix}; Y_{m-1} = \begin{pmatrix} y_{n-1} \\ y_n \end{pmatrix}; F_{m-1} = \begin{pmatrix} f_{n-1} \\ f_n \end{pmatrix}; F_m = \begin{pmatrix} f_{n+1} \\ f_{n+2} \end{pmatrix}$ in both cases.

Substituting the scalar test equation $y' = \lambda y$ ($\lambda < 0, \lambda$ is complex) into (12) and taking $\lambda h = \bar{h}$ gives

$$A_0 Y_m = A_1 Y_{m-1} + \bar{h}(B_0 F_{m-1} + B_1 F_m) \tag{13}$$

The stability polynomials of (10) and (11) are obtained by evaluating

$$R(t;\bar{h}) = Det\left[(A_0 - \bar{h}B_1)t - (A_1 + \bar{h}B_0)\right] = 0 \tag{14}$$

which respectively give

$$R(t;\bar{h}) = t^2 - \frac{175}{152}t^2\bar{h} - \frac{37}{38}t + \frac{25}{76}t^2\bar{h}^2 - \frac{67}{76}t\bar{h} - \frac{1}{38} - \frac{1}{76}t\bar{h}^2 - \frac{3}{152}\bar{h} = 0 \tag{15}$$

and

$$R(t;\bar{h}) = t^2 + \tfrac{108}{323}\bar{h}^2 t^2 - \tfrac{3}{323}\bar{h}^2 t - \tfrac{375}{323}\bar{h}t^2 - \tfrac{555}{646}\bar{h}t - \tfrac{11}{646}\bar{h} - \tfrac{317}{323}t - \tfrac{6}{323} = 0 \tag{16}$$

Zero stability of the methods is shown by setting $\bar{h} = 0$ in (15) and (16) to respectively yield

$$R(t;\bar{h}) = t^2 - \frac{371}{3838}t - — = 0 \tag{17}$$

$$R(t;\bar{h}) = t^2 - \tfrac{317}{323}t - \tfrac{6}{323} = 0 \tag{18}$$

Solving (17) and (18) respectively give the roots $t = 1$, $t = -\,0.02631578947$ and $t = 1$, $t = -\,0.01857585139$. Hence, the methods (10) and (11) are both zero stable.

The stability regions of (10) and (11) are determined by letting $t = e^{i\theta}$ in (15) and (16) and the resulting equations are plotted using MATLAB environment as indicated in Figures 1 and 2.

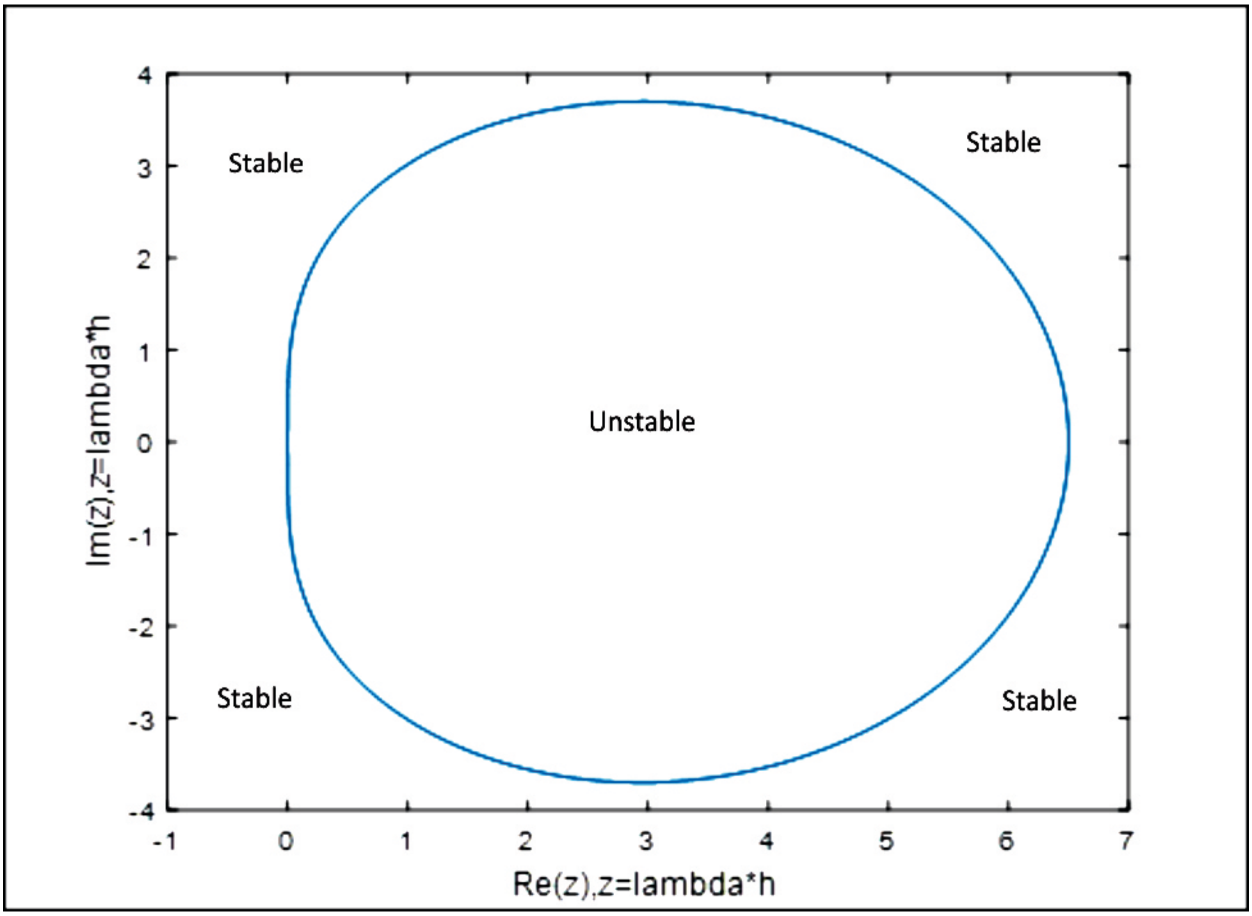

Figure 1: *Absolute stability region of I2BBDF2*

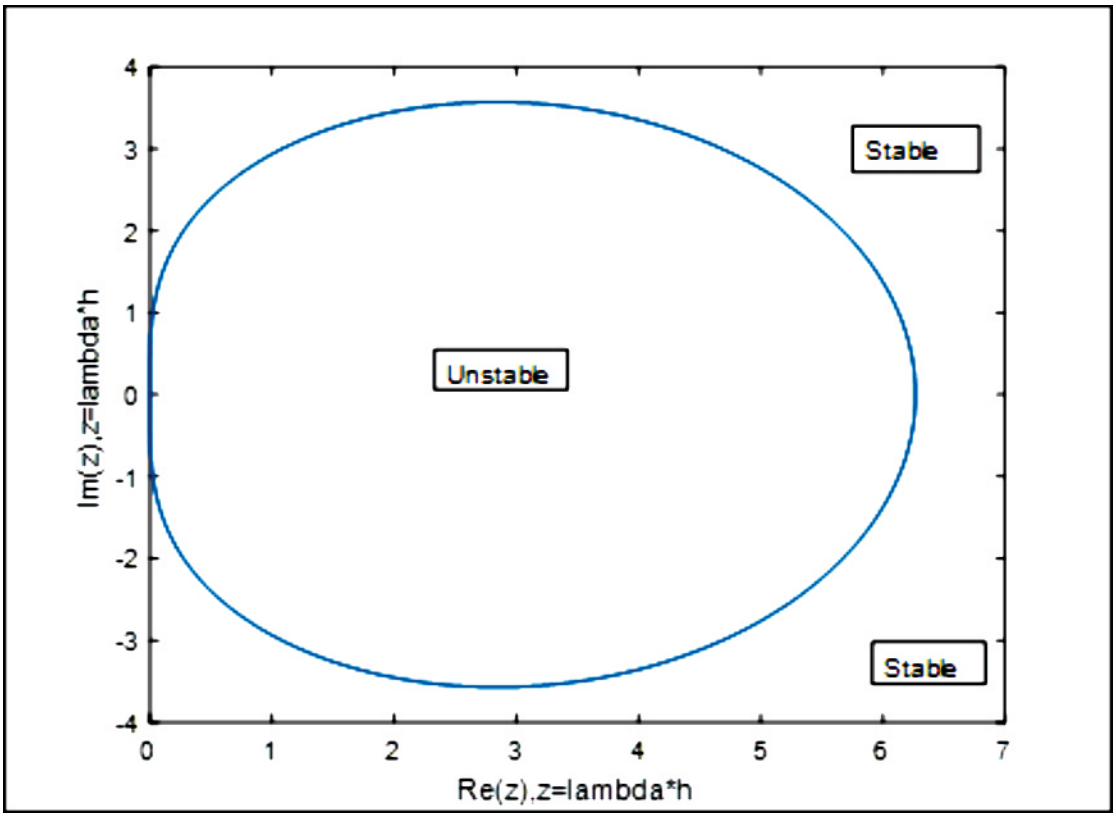

Figure 2: *Stability region of the I22BBDF2*

Figures 1 and 2 both show that the region of absolute stability contains the entire left half of the complex plane for methods (10) and (11) indicating that the methods are absolutely stable.

This is in contrast to the existing method of Mohamad *et al.,* (2018) whose absolute stability region implies the method is not wholly A-stable.

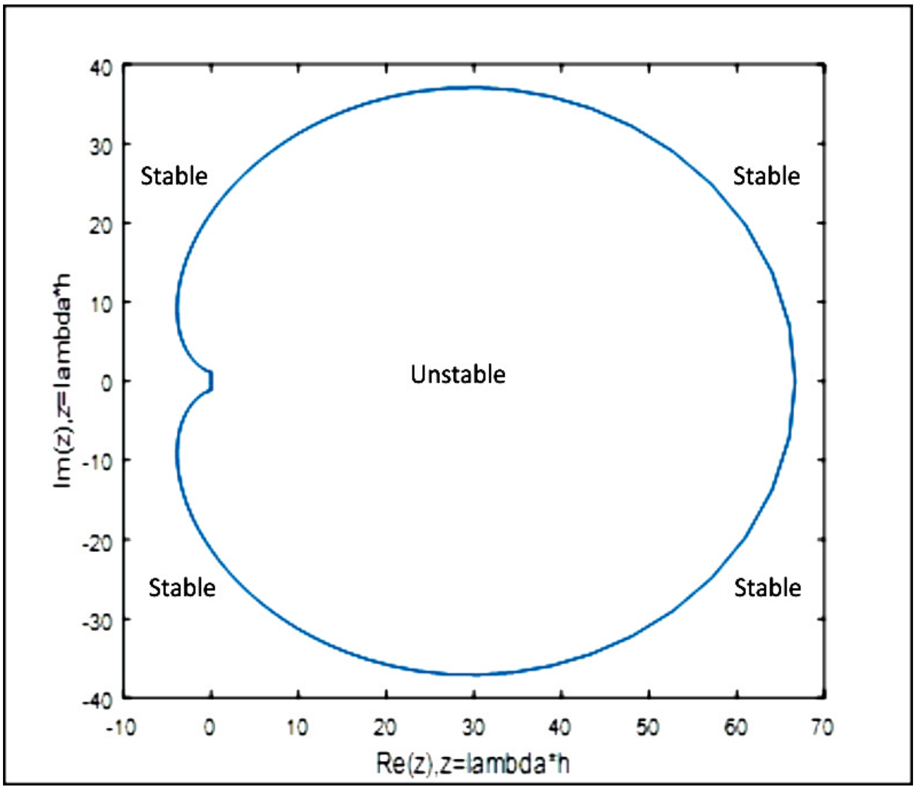

Figure 3: *Stability region of I2BBDF(5) by Mohamad et al., (2018).*

We present figure 4 to show the comparison between existing method by Mohamad *et al.,* (2018) and the new I2BBDF2 formula (10) in terms of the region of absolute stability.

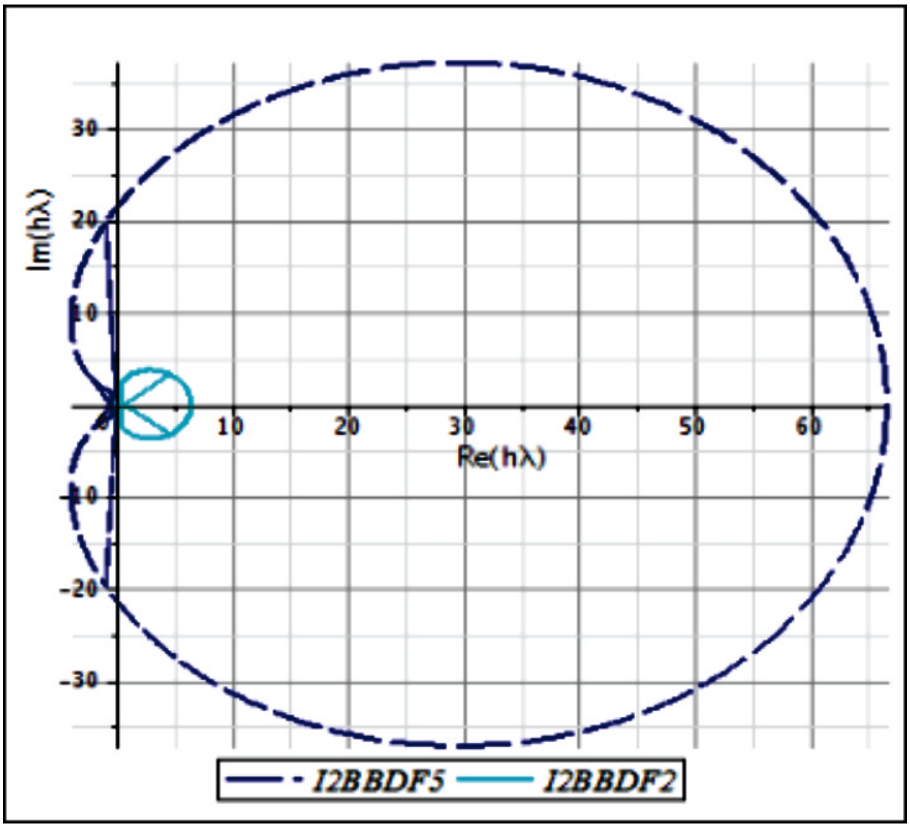

Figure 4: *Comparison of the stability regions of I2BBDF2 (10) and method by Mohamad et al., (2018).*

Order and Error Constants of the New Methods.

The error constant of the new method (10) is $C_3 = \begin{pmatrix} -\frac{3}{16} \\ 0 \end{pmatrix}$ while that of method (11) is

$C_3 = \begin{pmatrix} -\frac{11}{57} \\ 0 \end{pmatrix}$, implying that both methods are of order $p = 2$ .Thus, consistency of the methods is established.

Convergence of the Methods

Since the methods I2BBDF2 (10) and I22BBDF2 (11) are consistent and zero stable, by a well-known theorem, we have established their convergence which is a minimum requirement that every linear multistep method (LMM) must possess.

Implementation of the Methods

This section discusses the implementation of the methods by first provide a starting value using the fourth order Runge -Kutta method and afterwards, y_{n+1} and y_{n+2} values are generated simultaneously for methods (10) and (11) respectively. Thus, (10) and (11) are not self -starting methods.

Definition 3.4: Let y_i and $y(x_i)$ be the approximate and exact solution of(1) respectively, then the maximum error is evaluated by using the formula:

$$\mathrm{MAXE} = \max_{1 \le t \le NS} \left| (y_i)_t - (y(x_i))_t \right|$$

where, NS is the total number of steps.

Test Problems and Numerical Results

The following stiff initial value problems are used to test the performance of the methods.

Problem 1: [Mohamad *et al.,* (2018)]

$y' = -20y + 20\sin x + \cos x, \; y(0) = 1, \; 0 \le x \le 2$

Exact solution: $y(x) = \sin x + e^{-20x}$

Problem 2: [Muktar and Majid, (2017)]

$y' = -10y + 10, \; y(0) = 2, \; 0 \le x \le 10$

Exact solution: $y(x) = 1 + e^{-10x}$

Problem 3: [Musa *et al.,* (2013b)**]**

$y' = 100(\sin x - y), \; 0 \le x \le 3$

Exact solution: $y(x) = \dfrac{\sin x - 0.01\cos 0.01x + e^{-100x}}{1.0001}$

Tables 1 to 3 below show the results from applying the new methods (10) and (11) with comparison to some existing numerical methods in terms of maximum error. The following notations interpret the elements in the tables:

*h:*step size

BBDF(5): Fifth order block backward differentiation formula by Nazir *et al.,* (2012)

I2BBDF(5): Improved two-point block backward differentiation formula of order five by Mohamad *et al.,* (2018)

E2OSB(4): Extended two point one-step block method of order four by Muktar and Majid (2017)

2IBBDF: 2-Point block backward differentiation by Musa *et al.,* (2013b).

I2BBDF2: Proposed improved two-point BBDF of order two.

I22BBDF2: Proposed improved two-point BBDF of order two.

NS: Number of steps taken
MAXE : Maximum error

Table 1: Numerical results for problem 1

Step-size (h)	Method	NS	MAXE
	BBDF(5)	1000	9.71195e-04
10^{-3}	I2BBDF(5)	1000	7.35546e-04
	I2BBDF2	1000	2.00000e-09
	I22BBDF2	1000	5.80004e-09

Table 2: Numerical results for problem 2

Step-size (h)	Method	NS	MAXE
10^{-1}	E2OSB(4)	50	4.24e-003
	I2BBDF2	50	1.00e-009
	I22BBDF2	50	2.75e-043
10^{-2}	E2OSB(4)	500	2.48e-007
	I2BBDF2	500	8.00e-009
	I22BBDF2	500	4.00e-009

Table 3: *Numerical results of problem 3*

Step-size (h)	Method	NS	MAXE
	2IBBDF	150	1.82886e-004
10^{-2}	I2BBDF2	150	8.53350e-006
	I22BBDF2	150	7.83327e-007
	2IBBDF	1500	1.07874e-004
10^{-3}	I2BBDF2	1500	5.92400e-007
	I22BBDF2	1500	6.72700e-009

Discussion

From tables 1-3, it can be seen that the new methods (I2BBDF2 and I22BBDF2) outperformed the existing methods -E2OSB(4), 2IBBDF, BBDF5 and I2BBDF5 in terms of maximum errors respectively. Convergence is evident as the maximum error reduces as the step length of the methods tends to zero, specifically in Table 2 and 3. Thus, the maximum error indicates that the approximate solutions tend to the exact solution as the iteration processes continue. Notice that in problem 1, I2BBDF2 performed better than I22BBDF2 in terms of maximum error. This also indicates that the approximate solution for I2BBDF2 got closer to the exact solution than I22BBDF2 did. Also, for problem 3, the reverse was the case as I22BBDF2 outperformed I2BBDF2. Hence, the new methods converge faster than the existing methods on the respective problems considered.

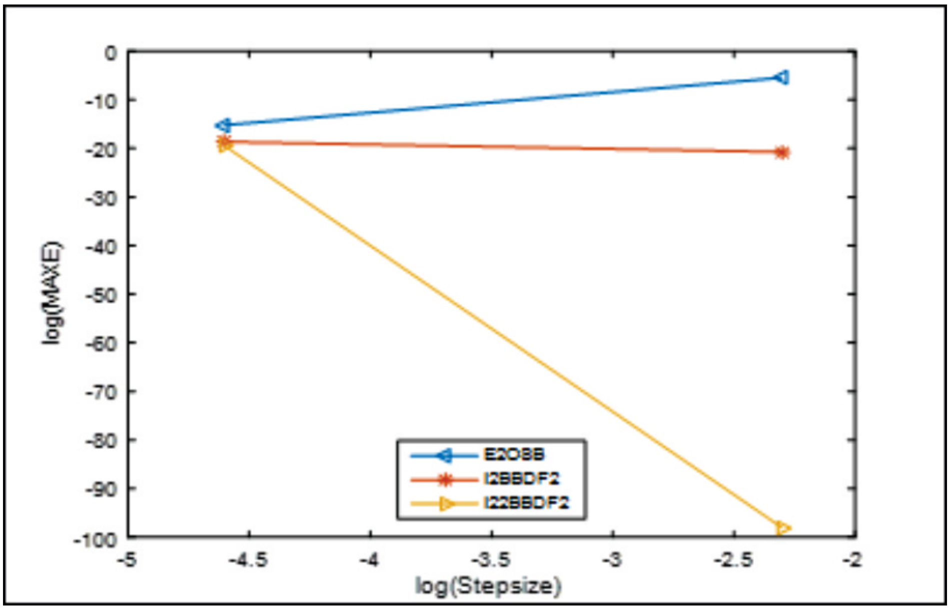

Figure 5: *Comparison of efficiency curves in terms of error for problem 2*

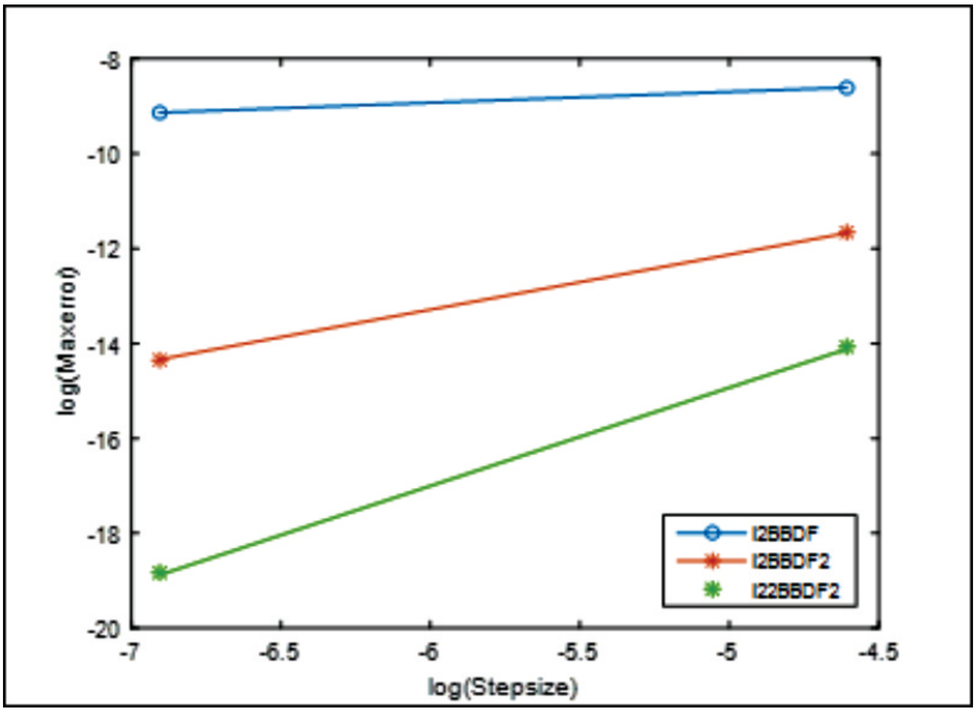

Figure 6: *Comparison of efficiency curves in terms of error for problem 3*

Figures 5 and 6 above also show that the scaled error for the new methods (10) and (11) are smaller when compared to that of the existing methods considered.

Conclusion

Two new methods called improved two-point block backward differentiation formula each of order two (I2BBDF2 and I22BBDF2) have been developed. Though both methods are of lower order two, it has been established that these are suitable for solving first order stiff initial value problems of ordinary differential equations. Comparison between the methods and existing methods showed that the new methods outperformed the existing methods. However, it has been observed that the I2BBDF2 method outperformed the I22BBDF2. This may due to the fact that former method has smaller unstable region than the latter. Hence, a wider region of absolute stability could result to the higher performance of the method in terms of maximum error.

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