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Application of the Schwarz-Christoffel Transformation to the Solution of Harmonic Dirichlet Problems in Electrostatics

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Abstract

In this paper, a purely conformal mapping method for efficiently solving harmonic Dirichlet problems of electrostatic in domains free of charge and with charge whose boundaries have inconvenient geometries consisting of straight line segments is presented. The method which uses the inverse of an appropriately determined Schwarz-Christoffel transformation as the mapping function, was applied to harmonic Dirichlet problems in an infinite strip and infinite sector and the solution or electrostatic potential for the problem obtained for each case. Furthermore, the equipotential lines of the electric field were also obtained in order to show the features of the solution and the field analysed accordingly. The electric field intensity was also analysed to show its variation in the field. This method could therefore be a suitable alternative method for solving Laplace's equation in two dimensional electrostatic problems.

Key words and Phrases: Conformal Map, Schwarz-Christoffel Map, Analytic Function, Branch of a Multiple Valued Function, Electrostatic Potential, Electric field Intensity.

Introduction

Any distribution of charge be it continuous, discrete, or a combination thereof gives rise to an electric field. The electric field intensity ε at a point Q in such a field is by definition the vector representing the force exerted on a unit positive charge placed there. This force is derivable from a scalar potential function called the electrostatic potential and is expressed mathematically by Spiegel (1974) as

$$\boldsymbol{\varepsilon} = -\nabla\phi \tag{1}$$

The problem of determining the electrostatic potential of an electric field in a domain D containing no charge requires solving a second order linear differential equation with constant coefficients

$$\nabla^2\phi = 0 \tag{2}$$

called Laplace's equation subject to some specified conditions on the boundary ∂D of D depending on the problem in question. When the values of ϕ are specified along the boundary of the domain the problem is called a Dirichlet problem and it is the concern of this paper. It is well known in the theory of analytic functions of a complex variables that if a function $f(z) = \phi(x,y) + i\psi(x, y)$ is analytic in a domain D then its component functions ϕ (x, y) and ψ (x, y) are harmonic there. The solution to problem (2) using complex variable methods therefore requires finding the real and imaginary parts of a function which is analytic in D and such that these component functions satisfy the boundary conditions. It is also well known that if a function ϕ (x, y) is harmonic throughout a domain D is harmonic throughout a domain ψ (x, y) conjugate to ϕ (x, y) such that the function $\Omega(z) = \phi(x, y) + i\psi(x, y)$ is analytic throughout D. In electrostatics the function Ω is called the complex electrostatic potential and is related to the electric field intensity ε by the formula

$$\boldsymbol{\varepsilon} = -\overline{\Omega'} \tag{3}$$

given by Spiegel (1974). The complex variable method of conformal mapping is a useful intermediate step in the solution and analysis of two dimensional harmonic Dirichlet problems in electrostatics as well as other Dirichlet problems in ideal fluid flows, electromagnetism, and thermal physics as is evident in the works of Churchill and Brown (1984), Spiegel (1974), Tobin and Lloyd (2002), Ganzolo *et al*. (2008), Tao *et al.* (2008), Anders (2008), Weiman *et al.* (2016), Yariv and Sherwood (2015), Andreas and Yorgos (2004), Xu *et al.* (2015), Wesley *et al*. (2008), Suman (2008) . The technique involves the transformation of the problem from a domain with an inconvenient geometry in one complex plane into a domain with a simpler geometry in another complex plane by means of an appropriate mapping function which preserves the magnitude of the angles between curves as well as their orientation. Amongst a variety of conformal transformations, the two most commonly used in the solution of electrostatic problems are the bilinear transformation and the Schwarz-Christoffel map. In this paper, we shall focus on the Schwarz-Christoffel map only. This transformation which is given by Churchill and Brown (1984) as

$$w = f(z) = A\int_{z_0}^{z}\prod_{j=1}^{n-1}(s - x_j)^{-k_j}ds + B \tag{4}$$

or

$$\frac{dw}{dz} = f'(z) = A\prod_{j=1}^{n-1}(z - x_i)^{-k_j} \tag{5}$$

is one that conformally maps the upper half Im $z > 0$ of the z plane and the entire x axis except for a finite number of points $x_1, x_2, ..., x_{n-1}, \infty$ in a one-to-one correspondence onto the interior of a given simple closed polygon and its boundary, respectively, such that $w_j = f(x_j)$ $(j = 1,2, ..., n-1)$ and $w_n = f(\infty)$ are the vertices of the polygon. The points $z = x_j$ $(j = 1,2, ..., n - 1)$ are arranged such that the order relation $x_1 < x_2 < ... < x_{n-1}$ is satisfied. The complex constants A and B in formula (4) determine the size, orientation and position of the polygon, the $k_j^{\prime}s$ are real constants between - 1 and 1 determined from the relation $-\pi < k_j\pi < \pi$, where $k_j\pi$ $(j = 1,2, ..., n-1)$ are the exterior angles at the vertices w_j $(j = 1,2, ..., n - 1)$ of the polygon, while the limits of integration z_0 and z are respectively fixed and variable points in the region Im $z \geq 0$ of analyticity of the Schwarz-Christoffel function. In order to make the function $f'(z)$ in (5) analytic everywhere in the region Im $z \geq 0$ except at the $n - 1$ points $z = x_j$ $(j = 1,2, ..., n - 1)$, we introduce branch lines or cuts extending below those points and normal to the real axis and write

$$(z-x_j)^{-k_j} = |z-x_j|^{-k_j} e^{-ik_j\theta_j} \left(|z-x_j| > 0, -\frac{\pi}{2} < \theta_j < \frac{3\pi}{2}\right) \quad 6$$

where $\theta_j = \arg(z-x_j)$ and $j = 1,2,\dots,n-1$ 1. It then follows that the function

$$G(z) = \int_{z_0}^{z} f'(z)dz \quad 7$$

is analytic in the region Im z = 0 and that $G'(z) = f'(z)$. Furthermore, the function $G(z)$ is defined at the points $z = x_j$ ($j = 1,2,\dots,n$ -1) such that it continuous there (Churchill and Brown 1984) so that the Schwarz-Christoffel transformation (4) is continuous throughout the region Im z = 0 and conformal there except for the points $z = x_j$ ($j = 1,2,\dots,n$ - 1).

In applications the domains usually encountered are simply connected (Churchill and Brown, 1984) and for such domains the existence of conformal maps is guaranteed by the Riemann mapping theorem which asserts that there exists a unique one to one conformal map from any simply connected domain D which is not the whole of the z plane onto the unit disc $|w| < 1$ in the w plane. It is also well known that if z_0 is any point in the upper half Im z > 0, then the bilinear transformation

$$w = e^{i\theta_0}\left(\frac{z-z_0}{z-\bar{z}_0}\right)$$

where θ_0 is a constant, conformally maps the upper half of the z plane in a one -to- one manner onto the unit disc $|w| < 1$ and conversely. Thus the Riemann mapping theorem also asserts that there exists a unique one-to-one conformal map from the upper half Im z > 0 of the z plane onto any simply connected domain which is not the whole of the z plane. Although the Riemann mapping theorem demonstrates the existence of a mapping function, it does not produce it.

However, for maps of the upper half Im z > 0 of the z plane onto the interior of a polygon the Schwarz -Christoffel formula provides explicit formulae that work. In this research paper, we shall apply the transformation in the solution of Dirichlet harmonic problems of electrostatics posed in domains consisting of straight line segments for the cases of an infinite strip and infinite sector of angle α $(0 < \alpha < \frac{\pi}{2})$

Methodology

Consider an electric field in a domain D of the plane containing no charge and due to constant but different potentials on the parts of its boundary ∂D which consists of straight line segments. The problem of determining the electrostatic potential inside D requires solving the mathematical problem (2) subject to some conditions on the boundary ∂D for which ϕ takes prescribed values. In order to solve this problem the specific Schwarz - Christoffel transformation $w = f(z)$ that maps the upper half Im z > 0 of the z plane in a one-to one manner onto Ω which satisfies the boundary conditions

$$w_j = f(x_j) \;\; (j = 1,2,\dots,n-1) \text{ and } w_n = f(\infty)$$

where $x_1 < x_2 < \cdots < x_{n-1}, x_n = \infty$

is first determined from the generalized form of the transformation (4) or (5). Solving for z in terms of w, the inverse function $z = g(w)$ which transforms the problem domain D and hence the given electric field in the w plane onto one in the upper half Im z > 0 of the z plane is then obtained. If the inverse map turns out to be multiple valued, then it is made single valued and analytic everywhere in D using the complex variable method of branch cuts or lines. The various portions of the boundary ∂D of D with their respective potentials are also mapped by the inverse Schwarz-Christoffel transformation onto the appropriate portions of the x axis, respectively. We note here that the transformation of a harmonic function via a conformal map remains harmonic (Spiegel, 1974). The inverse function thus simplifies the given harmonic Dirichlet problem to one in the upper half Im z > 0 of the z plane and which satisfies the boundary conditions on the x axis of the type shown in figure 1.

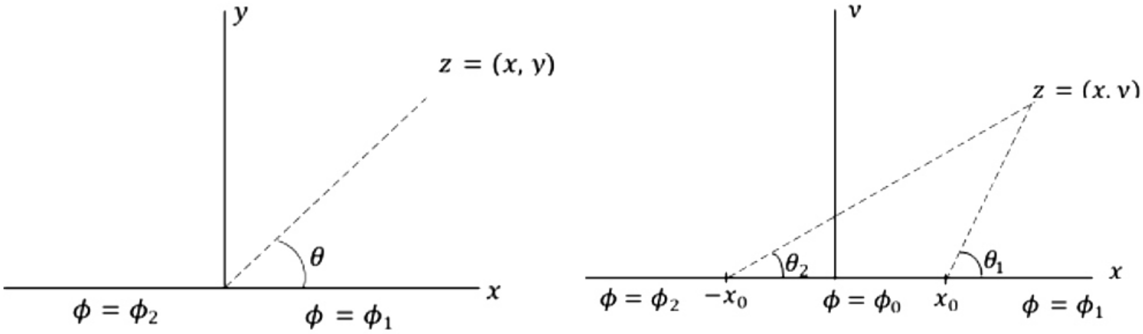

Figure 1: The Two Types of Boundary Conditions on the *x* axis

In the first diagram of figure 1, the function $\phi = A\theta_1 + B$ where A and B are real constants is harmonic in the upper half Im z > 0 of the z plane since it is the imaginary part of the function $f(z) =$ Aln z +iB, where the branch of ln z is given as

$$\ln z = \ln r + i\theta \ (r > 0, -\frac{\pi}{2} < \theta < \frac{3\pi}{2}) \qquad 8$$

The constants A and B are then determined using the boundary conditions along the x axis

$$\phi = \phi_1 \text{when } x > 0; \text{i.e } \theta = 0 \text{ and } \phi = \phi_2 \text{when } x < 0; \text{i.e } \theta = \pi$$

to obtain the solution or the electrostatic potential of the problem in the z plane. The required solution in the w plane is then obtained using the inverse function $z = g(w)$ by replacing u and v for x and y respectively in the expression for $\phi(x, y)$. In the second case in which the boundary condition is given in the second diagram of Figure1, the harmonic function in the upper of the z plane is $\phi = A\theta_1 + B\theta_2 + C$ since it is the imaginary part of the function $f(z) =$ $A\ln(z - x_0) + B\ln(z + x_0) + iC$. Here too the real constants A, B and C are determined using the boundary conditions and the solution ϕ is determined as before.

Alternatively, the transformed boundary value problem in the upper half Im z > 0 of the z plane can be solved using Poisson's integral formula for that domain.

If the problem domain D contains a charge q per unit length at $z = z_0$ and the line charge -q per unit length at $z = \bar{z}_0$ is given as

$$\Omega(z) = 2q \ \ln(z - \bar{z}_0) - 2q \ \ln(z - z_0) = 2q \ln\left(\frac{z - \bar{z}_0}{z - z_0}\right) \qquad 9$$

The electrostatic potential due to the line charge q per unit length at $z = z_0$ and the plat plate at zer o potential in the z plane is therefore

$$\phi = Re\left[2q \ln\left(\frac{z - \bar{z}_0}{z - z_0}\right)\right] \qquad 10$$

Thus the required electrostatic potential at any point in the domain D is found from the inverse function by replacing z with $g(w)$ in expression (10). The equipotential lines of the electric field are then generated by setting the electrostatic potential to a constant and then varying its values.

Results

In this section we present the solution of some harmonic Dirichlet problems in electrostatics posed in an i nfinite strip and infinite sector using the purely conformal based method outlined in the methodology.

Problem 1: (Electrostatic Potential in an Infinite Strip of Width α)

Problem 1(a) .We first consider the harmonic Dirichlet problem in equation (2) for the determination of the electrostatic potential ϕ (u, v) between two parallel conducting plates of infinite extent in which line charge q per unit length at $z = z_0$ and the boundaries are grounded, then, we use the fact that the electrostatic potential due to a line charge q per unit length at the point $z = z_0$ parallel to a flat plate at potential zero is the same as replacing the plate with the line charge $-q$ at $z = \bar{z}_0$ (Spiegel, 1974). The complex electrostatic potential due to the line the region between them is represented in two dimensions by the infinite strip in figure (2)gt and described by the equation

$0 < v < a, \quad -\infty < u < \infty$

The part of the boundary of the strip corresponding to the line $v = 0$ (that is, the u axis of the w plane) is kept at potential $\phi = 0$ while the part corresponding to the line $v = \alpha$ is kept at potential $\phi = 1$.

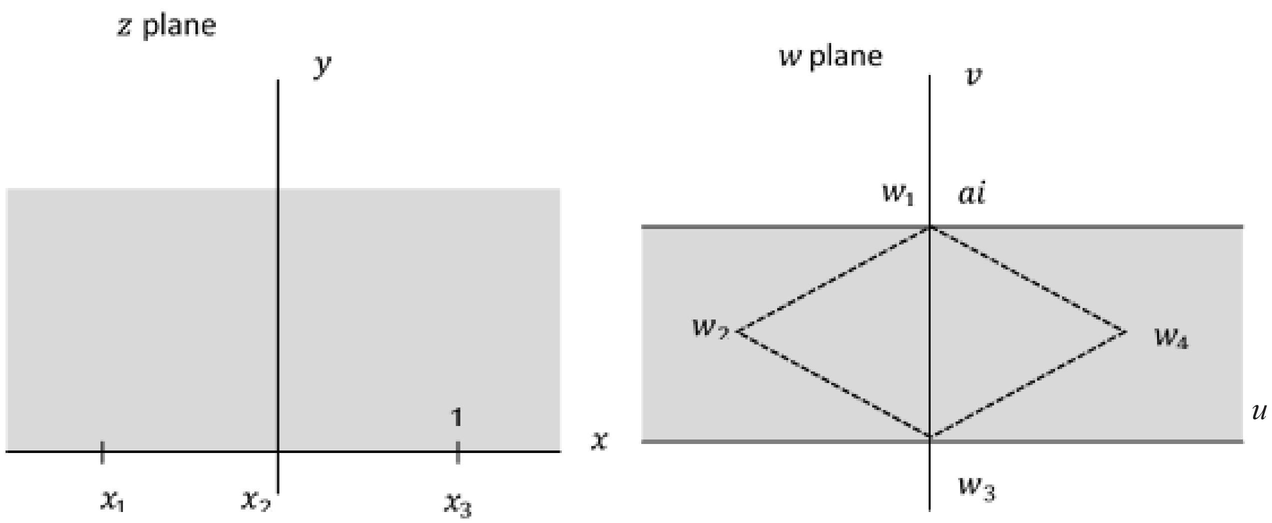

Figure 2: One-to-one mapping of the upper half Im z > 0 of the z plane onto an infinite strip in the w plane.

The Schwarz-Christoffel transformation $w = f(z)$ that maps the half plane $\text{Im } z > 0$ and the entire real axis except the origin in a one - to - one manner onto the strip and its boundary, respectively was found to be

$$w = \frac{a}{\pi} \ln z \quad (|z| > 0, 0 \leq \arg z \leq \pi \qquad 11$$

by considering the strip as a limiting form of a rhombus represented by the dashed line in figure 2 with vertices at $w_1 = ai, w_2, w_3 = 0,$ and $w_4 = \infty$ respectively or using the table of transforms given by Spiegel (1974) and Churchill and Brown (1984). In this problem, the point x_1 is to be determined while the values $x_2 = 0, x_3 = 1,$ and $x_4 = \infty$ are given. The value of x_1 was found to be - 1. The inverse transformation is therefore

$$z = e^{\frac{\pi}{a}w} = g(w) \qquad 12$$

and maps the strip in a one -to-one manner onto nonzero points in the upper half plane $\text{Im } z = 0$. The part of the boundary of the strip corresponding to the line $v = 0$ at zero potential is mapped by the transformation (12) onto the positive real axis $x > 0$ at potential zero while the part corresponding to the line $v = a$ at unit potential maps onto the negative real axis $x < 0$ in the z plane at potential unity. The given harmonic Dirichlst problem then simplifies to one in the upper half $\text{Im } z > 0$ of the z plane subject to the boundary conditions: $\phi = \phi_1 = 1$ when $x > 0;\ \theta = 0$ and $\phi = \phi_2 = 0$ when $x < 0;\ \theta = \pi$ of the first type in figure 1(a). The function $\phi = A\theta + B,$ where A and B are real constants is harmonic in the upper half $\text{Im } z > 0$ of the z plane since it is the imaginary part of the analytic function $A \ln z + Bi$. The values of the real constants A and B were found to be $1/\pi$ and 0 respectively. Hence, the electrostatic potential in the z plane is

$$\phi = \frac{1}{\pi}\theta = \frac{1}{\pi}\tan^{-1}\left(\frac{y}{x}\right)$$

and the required solution or electrostatic potential in the strip is therefore

$$\phi(u,v) = \frac{v}{a} \quad (13)$$

where

$$x = e^{\frac{\pi u}{a}}\cos\left(\frac{\pi v}{a}\right) \text{ and } y = e^{\frac{\pi u}{a}}\sin\left(\frac{\pi v}{a}\right)$$

from equation (12). On setting equation (13) equal to a constant c $(0 \le c \le 1)$, we obtain the equipotential lines

$$v = ac \quad (14)$$

The electric field intensity at any point w in the strip is

$$\varepsilon = -\frac{i}{a} \quad (15)$$

and has magnitude as

$$|\varepsilon| = \frac{1}{a} \quad (16)$$

Problem 1 (b). Now consider a situation in which the plates in problem 1(a) are both kept at zero potential and a parallel semi-infinite plate, placed midway between them, is kept at unit potential as shown in figure 3(a). We now determine the electrostatic potential in the region between those plates.

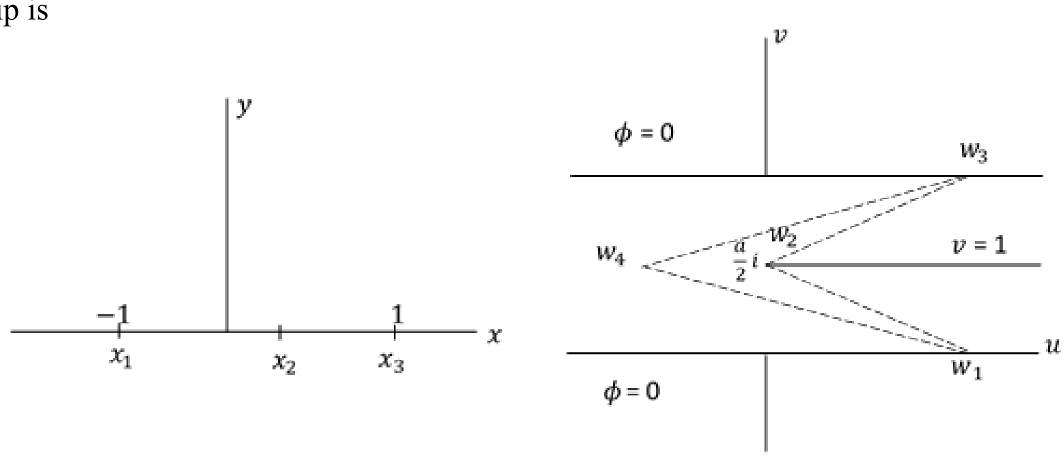

Figure 3(a): One-to-one mapping of the upper half $\text{Im } z > 0$ of the z plane onto the Interior of an infinite strip With a Semi-Infinite Strip Placed at $w = \frac{a}{2}i$ in the w plane.

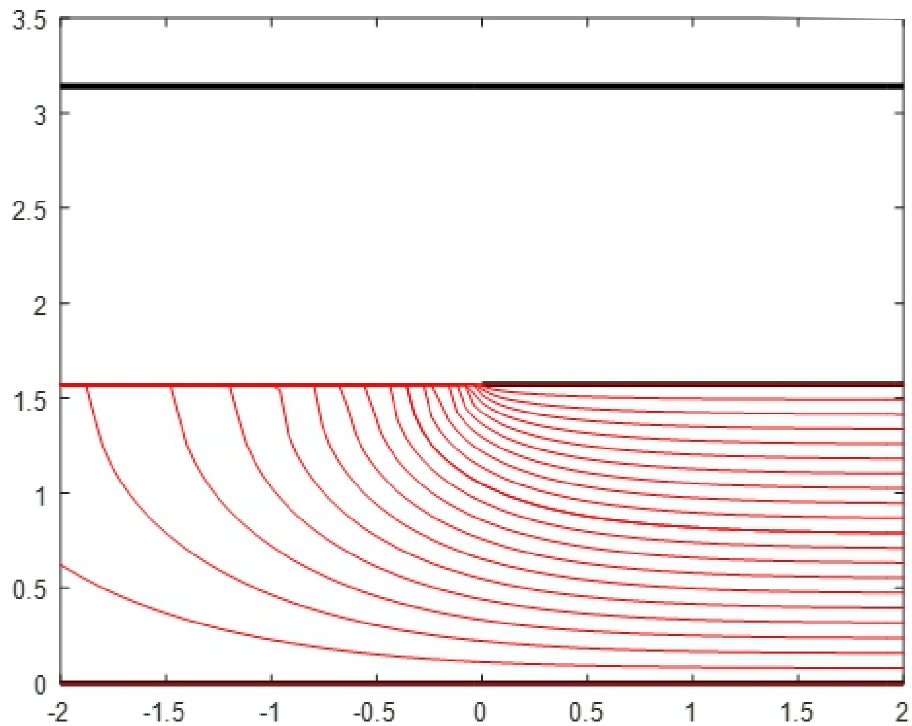

Figure 3(b): Plots of Equipotential Lines Interior to an infinite strip of Width π units With a Semi-Infinite Strip Placed at $w = \frac{\pi}{2}i$ in the w plane.

Proceeding from the conformal mapping aspect of the solution in part (a) of problem (1), the inverse of the Schwarz - Christoffel transformation (12) is first applied to transform the given problem to a corresponding simpler one in the upper half of the z plane. The line charge q at the point $w = bi$ is mapped by the transformation (12) into a line charge q at the point $z_0 = e^{\frac{\pi bi}{a}}$ To find the electrostatic potential due to the line charge q per unit length at the point $z_0 = e^{\frac{\pi bi}{a}}$ parallel to the flat plate at zero potential, we simply replace the flat plate with the line charge - q per unit length at $z_0 = e^{-\frac{\pi bi}{a}}$. The complex electrostatic potential due to the line charge q per unit length at $z_0 = e^{\frac{\pi bi}{a}}$ and the line charge - q per unit length at $z_0 = e^{-\frac{\pi bi}{a}}$ is given as

$$z_0 = e^{-\frac{\pi bi}{a}} \tag{28}$$

Hence, the electrostatic potential at any point in the upper half $\operatorname{Im} z > 0$ of the z plane

$$\phi = Re\left[2q\ln\left(\frac{z - e^{-\frac{\pi bi}{a}}}{z - e^{\frac{\pi bi}{a}}}\right)\right] \tag{29}$$

Thus the required potential at any point in the strip is

$$\phi = Re\left[2q\ln\left(\frac{e^{\frac{\pi w}{a}} - e^{-\frac{\pi bi}{a}}}{e^{\frac{\pi w}{a}} - e^{\frac{\pi bi}{a}}}\right)\right] \tag{30}$$

on letting $z = e^{\frac{\pi}{a}w}$ in equation (29).

The next problem is motivated by the fact that the technique used is often employed to obtain closed form expressions for the characteristic independence and dielectric constant of different types of waveguides (Suman, 2008).

Problem 2: (Electrostatic Potential in an infinite Sector of Angle α)

Problem 2(a). Now consider the harmonic Dirichlet problem in equation (2) for the determination of the electrostatic potential in an angular sector bounded by two infinite plane conductors inclined at angle α $(0 < \alpha < \frac{\pi}{2})$ and charged to constant potentials ϕ_1 and ϕ_2, respectively as shown in Figure 4(a).

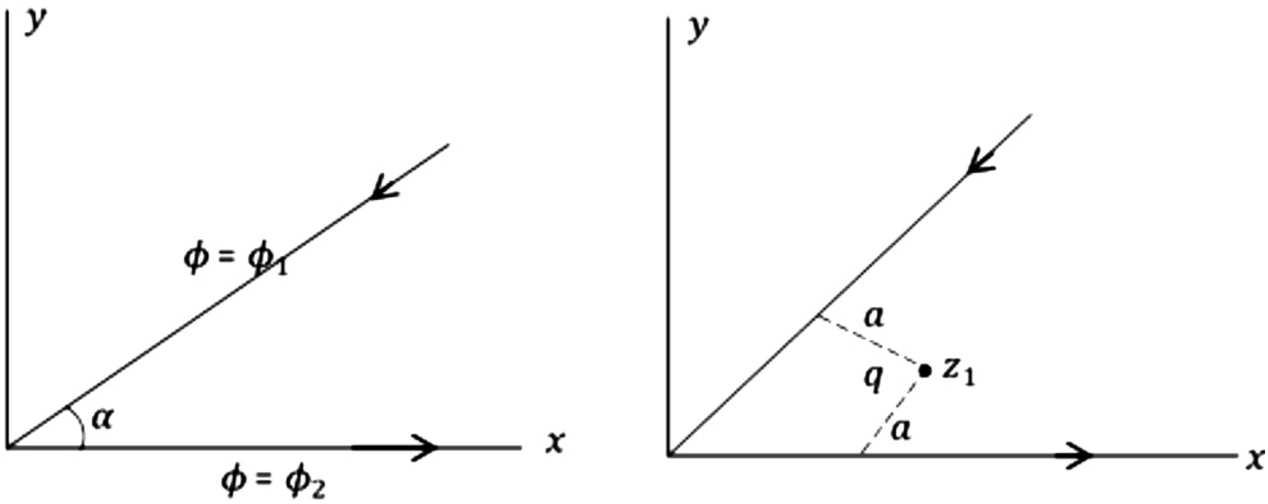

Figure 5: Infinite Sector of Angle α $(0 < \alpha < \frac{\pi}{2})$ Without Charge and With Charge at z_1.

The Schwarz-Christoffel transformation that maps the upper half $\mathrm{Im}\, z \geq 0$ of the z plane in a one -to one manner onto the infinite sector $|w| \geq 0, 0 \leq \arg w \leq m\pi$ $(0 < m < 1)$ in the w plane such that the point $z = 1$ is mapped into $w = 1$ is given by Churchill and Brown (1984) and Spiegel (1974) as

$$w = z^m \ (0 < m < 1) \tag{31}$$

where the angular region is considered as the limiting form of a triangle as the angle τ tends to zero. We note here that the angle α between the plates in the given problem is a particular case of the infinite sector with α $= \pi m$ where $0 < m < \frac{1}{2}$. The inverse function is therefore

$$z = w^{\frac{1}{m}} = g(w) \tag{32}$$

and maps the infinite sector in a one -to-one manner onto the upper half Im z = 0 of the z plane. If $w = \rho e^{i\sigma}$ $(0 \leq \sigma \leq m\pi)$ and $z = re^{i\beta}$ then $r = \rho^{\frac{1}{m}} > 0$ and $\beta = \frac{\sigma}{m}$

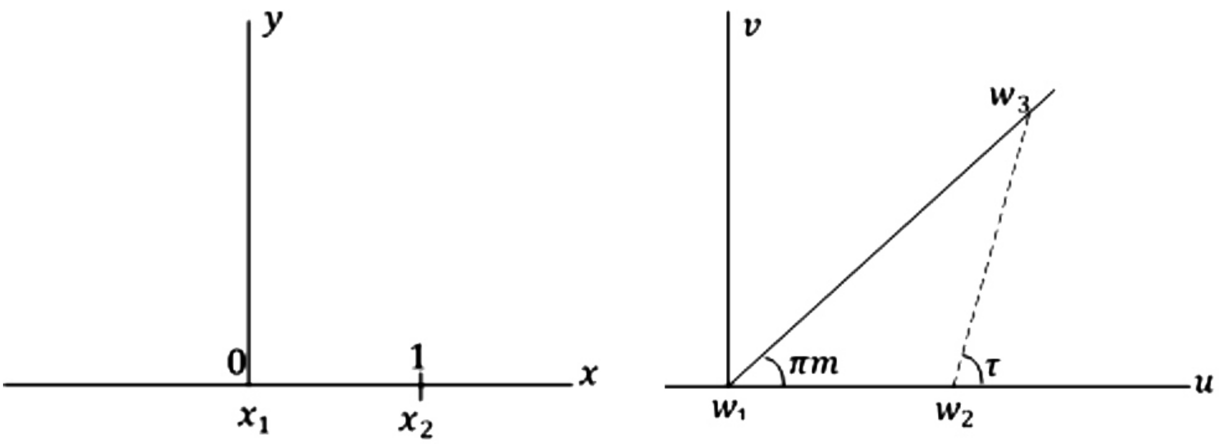

Figure 6: One-to-one mapping of the upper halfIm z > 0 of the z plane onto an infinite Sector in the w plane.

Hence, the part of the boundary of the infinite sector corresponding to the positive real axis $u > 0$ of the w plane (i.e, th e ray $\rho > 0, \sigma = 0$) at potential ϕ_2 is mapped by the transformation (32) onto the positive real axis $x > 0$ of the z plane (the ray $r > 0, \beta = 0$) at potential ϕ_2 while the part corresponding to the ray $\rho > 0, \sigma = m\pi = \alpha$ at potential ϕ_1 maps onto the negative real axis $x < 0$ in the z plane(i.e; the ray $r > 0, \beta = \pi$) at potential Φ_1. The point $w = 0$ map into the point $z = 0$. The inverse transformation (32) therefore reduces the given Dirichlet problem to one in the upper half $\mathrm{Im}\, z > 0$ of the z plane subject to the following boundary conditions: $\phi = \phi_2$ when $x > 0$; $\theta = 0$ $\phi = \phi_1$ when $x < 0$; $\theta = \pi$. The constants A and B are determined using the boundary conditions to obtain their values as $A = (\phi_1 - \phi_2)/\pi$ and $B = \phi_2$. Hence

$$\phi = \phi_2 + \left(\frac{\phi_1 - \phi_2}{\pi}\right)\theta$$

or

$$\left(\frac{\phi - \phi_2}{\phi_1 - \phi_2}\right)\pi = \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Further simplification yields

$$\tan\left(\frac{\phi - \phi_2}{\phi_1 - \phi_2}\right)\pi = \frac{y}{x} = \tan\left(\frac{\sigma}{m}\right)$$

where

$$x = \rho^{\frac{1}{m}} \sin\frac{\sigma}{m} \text{ and } y = \rho^{\frac{1}{m}} \cos\frac{\sigma}{m}$$

and $w = \rho e^{i\sigma}$ $(0 \leq \sigma \leq m\pi)$ and $z = re^{i\beta}$

in the inverse function (32). Thus

$$\phi = \phi_2 + \left(\frac{\phi_1 - \phi_2}{m\pi}\right)\sigma = \phi_2 + \left(\frac{\phi_1 - \phi_2}{\alpha}\right)\theta \tag{33}$$

since both σ and θ vary from 0 to π . On setting the electrostatic potential in eq uation (33) equal to a constant, then the equation of the equipotential lines for the electric field is obtained as

$$\theta = \left(\frac{c-\phi_2}{\phi_1-\phi_2}\right)\alpha \quad \phi_1 \neq \phi_2 \qquad 34$$

or

$$y = x.\tan^{-1}\left[\left(\frac{c-\phi_2}{\phi_1-\phi_2}\right)\alpha\right] \quad \phi_1 \neq \phi_2 \qquad 35$$

The electric field intensity at any point in the field was found using the definition of the gradient of a scalar function in polar form

$$\boldsymbol{\varepsilon} = -\nabla\phi = -\left(\frac{\partial\phi}{\partial r}e_r + \frac{1}{r}\frac{\partial\phi}{\partial\theta}e_\theta + \frac{\partial\phi}{\partial z}e_z\right)$$

and its expression is

$$\boldsymbol{\varepsilon} = \frac{\phi_2-\phi_1}{\alpha r}e_\theta \qquad 35$$

The magnitude $|\boldsymbol{\varepsilon}|$ of the electric field intensity $\boldsymbol{\varepsilon}$ at any point in the electric field is therefore

$$|\boldsymbol{\varepsilon}| = \frac{|\phi_2-\phi_1|}{\alpha}.\frac{1}{r} \qquad 37$$

Problem 2(b). Suppose that in problem 2(a) the infinite plate conductors are now grounded and that a line charge q per unit length is located at point z_1 at equal distances α from the boundaries as shown in the second diagram of Figure 5. The task once again is to find the electrostatic potential in the angular region containing this charge.

Just like in the solution for part (a) of the problem, the inverse of the Schwarz-Christoffel transformation (32) is first applied to transform the given problem to a corresponding simpler one in the upper half of the z plane. The line charge q per unit length at the point $w_1 = |w_1|e^{i\theta_1}$ is mapped by the transformation (32) into a line charge q at the point $z_1 = |w_1|^{\frac{1}{m}}e^{\frac{\theta_1}{m}} = |w_1|^{\frac{\pi}{\alpha}}e^{\frac{\pi\theta_1}{\alpha}}$ where $\alpha = m\pi$ in the problem. We now apply the fact that electrostatic potential due to a line charge q per unit length at the point $z = z_0$ parallel to a flat plate at potential zero is the same as replacing the plate with the line charge - q at $z = \bar{z}_0$. First, the complex electrostatic potential due to the line charge q per unit length at $z_1 = |w_1|^{\frac{\pi}{\alpha}}e^{\frac{\pi\theta_1}{\alpha}}$ and the line charge - q per unit length at $\bar{z}_1 = |w_1|^{\frac{\pi}{\alpha}}e^{-\frac{\pi\theta_1}{\alpha}}$ is given as

$$\Omega(z) = 2q\,\ln\left(z - |w_1|^{\frac{\pi}{\alpha}}e^{-\frac{\pi\theta_1}{\alpha}}\right) - 2q\,\ln\left(z - |w_1|^{\frac{\pi}{\alpha}}e^{\frac{\pi\theta_1}{\alpha}}\right)$$

$$= 2q\ln\left(\frac{z - |w_1|^{\frac{\pi}{\alpha}}e^{-\frac{\pi\theta_1}{\alpha}}}{z - |w_1|^{\frac{\pi}{\alpha}}e^{\frac{\pi\theta_1}{\alpha}}}\right)$$

The electrostatic potential at any point in the upper half of the z plane is therefore

$$\phi = Re\left[2q\ln\left(\frac{z-\bar{w}_1^{\frac{\pi}{\alpha}}}{z-w_1^{\frac{\pi}{\alpha}}}\right)\right] \qquad (38) \text{ or}$$

$$\phi = \text{Im}\left[-2qi\left(\frac{z-w_1^{\frac{\pi}{\alpha}}}{z-\bar{w}_1^{\frac{\pi}{\alpha}}}\right)\right] \qquad (39)$$

on multiplying and dividing equation (34) by -1 and then multiplying it by the imaginary number i. Thus the electrostatic potential at any point in the strip is

$$\phi = \text{Im}\left[-2iq\ln\left(\frac{w^{\frac{\pi}{\alpha}}-w_1^{\frac{\pi}{\alpha}}}{w^{\frac{\pi}{\alpha}}-\bar{w}_1^{\frac{\pi}{\alpha}}}\right)\right] \qquad (40)$$

Discussion

Analysis of the Electric Field for the Various Cases of the Infinite Strip of Width ?? in Problem 1.

In analysing the electric field for problem 1(a), we first note from equation (13) that the electrostatic potential is indeed its solution since it satisfies Laplace's equation $\phi_{uu} + \phi_{vv} = 0$ and the boundary conditions $\phi(u,0) = 0$ and $\phi(u,a) = 1$. Clearly, the equipotential lines are straight lines which are perpendicular to each other and the u axis. We observe too that the boundary of the strip corresponding to the lines $v = 0$ and $v = a$ are also equipotential lines. The electrostatic potential is also constant along the equipotential lines. In particular, on the equipotential line $v = \frac{a}{2}$ corresponding to in equation (14) the $c = \frac{1}{2}$ electrostatic potential $\phi = \frac{1}{2}$, a constant. From equation (16), we see that the magnitude of the electric field intensity varies inversely with the plate width α. On the boundary $v = a = 0$ of the strip where $\phi = 0$, the magnitude $|\boldsymbol{\varepsilon}|$ of the electric field intensity tends to infinity, while $|\boldsymbol{\varepsilon}| = 1$ on the boundary $v = a = 1$ where $\phi = 1$

In problem 1(b) the boundaries of the infinite strip in problem 1(a) were then grounded and a semi -infinite strip at potential $\phi = 1$ was placed midway between the strip to obtain a new boundary value problem. Figure 3(b) show the equipotential lines generated by setting the strip width $a = \pi$. Here too the equipotential lines show paths in the electric field where the solution or electrostatic potential is constant.

The domains of problems 1(a) and 1(b) had no charge and so in part (c) of the problem, we introduced line a charge as additional factor affecting electrostatic potential. Figure 4 show MATLAB plots of equipotential lines due to the line charge per unit length placed at the point $b = \frac{\pi}{2}i$ in the strip of width $a = \pi$. On these lines the electrostatic potential is constant meaning that no work is required in moving a charge along any of those lines. Work is however needed in moving a charge from one of the equipotential lines to another.

Analysis of the Electric Field for the Two Cases of the Infinite Sectors of Problem 2.

Here too the electrostatic potential $\phi(r, \theta)$ in equation (33) is the solution of problem 2(a) since it satisfies the polar form of Laplace's equation

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = 0$$

where $x = r\cos\theta$ and $y = r\sin\theta$ and the boundary conditions $\phi(r, 0) = \phi_2$ and $\phi(r, \alpha) = \phi_1$. The equipotential lines are rays from the origin of the w plane as is clearly evident from equations (34) or (35). Observe too that the boundaries of the infinite sector are also equipotential lines of the field. From equation (37), it is clear that the magnitude of the electric field intensity at any point in the electric field is inversely proportional to its distance from the origin of the w plane for each fixed angle α.

In part (b) of the problem, we introduced line charges as additional factors affecting electrostatic potential and obtained the solution for this case too.

Conclusion

In this paper, a purely conformal mapping method for efficiently solving some harmonic Dirichlet problems of electrostatics in domains free of charge and then with charge is presented. Using this method the electrostatic potential of an electric field interior to and on the boundary of an infinite strip and infinite sector were determined and their equipotential lines analysed. We however note here that although this method gives exact analytical solutions and has interesting features, it is not without limitations. One such limitation of the method has to do with the evaluation of the integral involved in the Schwarz-Christoffel transformation and consequently recommend the use of numerical techniques in such situations (see paper by Thomas and Everett (2011). Secondly, because the method is purely conformal it is limited to two dimensional problems only and in particular to problems whose boundaries consist of straight line segments. We therefore suggest that further research in this field should focus on extending the work to include domains with other boundaries such as in cylinders.

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