One-Step Embedded Hybrid Block Method for Solving First Order Stiff Initial Value Problems of Ordinary Differential Equations

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ABSTRACT

We propose a new hybrid method by embedding the extended four-step backward differentiation formulae of Akinfenwa & Jator (2015) into a one-step method by a continuous approximation via multistep collocation technique for the solution of first-order stiff initial value problems of ordinary differential equations. The embedded hybrid block method (EHBM) here consists of four discrete formulae which are simultaneously used as integrators. Analysis of the properties of the EBHM indicate that the method is of order five, convergent, and A-stable making it suitable for solving stiff problems. Implementing the proposed method using some numerical examples shows its accuracy when compared with existing methods in the literature.

Keywords: Hybrid block method, Embedded BDF, Multistep collocation,.

INTRODUCTION

In this paper, we shall consider the first order initial value problem (IVP) in ordinary differential equations (ODE) of the form

$$y' = f(x, y(x)), \;\; y(x_0) = y_0 \tag{1}$$

on the interval $[x_0, x_N]$ and assume the existence of a unique solution of (1). Over the years, many authors have sought different numerical solutions for (1) especially when it is stiff in nature. Curtiss and Hirschfelder (1952) first introduced a class of multistep methods called the backward differentiation formulae (BDF) which became well-known for the solution of stiff IVPs because they possess infinite regions of absolute stability Over the years, several A-stable methods for solving (1) have been developed and can be found in the literature, (see Watts & Shampine, 1972; Musa *et al.*, 2013; Kumleng *et al.*, 2017; Yohanna, 2017; Nursyazwani & Zarina, 2019). Block methods as presented by Ibrahim & Nasarudin (2020) have some drawbacks that led to the introduction of hybrid methods.

These hybrid methods are known have increased accuracy and stability properties (Ezzeddine & Hojjati, 2011). Onumanyi *et al.*, (2001) reformulated the conventional BDF by embedding them into one-step hybrid methods which resulted in increased accuracy and improved stability properties of the method.

In this paper, we will construct a hybrid method by embedding the extended BDF formula (EBDF) of Akinfenwa and Jator (2015) for step number $k = 4$ into a one-step method similar to Onumanyi *et al.*, (2001) for the solution of first-order IVPs.

Derivation of the Proposed New Method

The general form of the EBDF of Akinfenwa and Jator (2015) is given as

$$y(t) = \sum_{j=0}^{k-1} \alpha_j(t) y_{n+j} + h\left[\beta_{k-1}(t) f_{n+k-1} + \beta_k(t) f_{n+k}\right] \tag{2}$$

where $\alpha_j(t),\ j = 0,\ldots,k-1,\ \beta_{k-1}(t)$ and $\beta_k(t)$ are continuous coefficients to be determined. Embedding (2) into a one-step method gives

$$y(t) = \sum_{j=0}^{\frac{k-1}{k}} \alpha_{\frac{j}{k}}(t) y_{n+\frac{j}{k}} + h\left[\beta_{\frac{k-1}{k}}(t) f_{n+\frac{k-1}{k}} + \beta_1(t) f_{n+1}\right] \tag{3}$$

When $k = 4$, (3) can be expressed explicitly as

$$y(t) = \alpha_0(t) y_n + \alpha_{\frac{1}{4}}(t) y_{n+\frac{1}{4}} + \alpha_{\frac{1}{2}}(t) y_{n+\frac{1}{2}} + \alpha_{\frac{3}{4}}(t) y_{n+\frac{3}{4}} + h(\beta_{\frac{3}{4}}(t) f_{n+\frac{3}{4}} + \beta_1(t) f_{n+1}) \tag{4}$$

where $\alpha_0(t), \alpha_{\frac{1}{4}}(t), \alpha_{\frac{1}{2}}(t), \alpha_{\frac{3}{4}}(t), \beta_{\frac{3}{4}}(t), \beta_1(t)$ are polynomial coefficients to be determined.

From (4), we consider an approximate solution $y(x)$ for solving (1) by a polynomial $p(x)$ in the form

$$y(x) = p(x) = \sum_{j=0}^{5} a_j x^j \tag{5}$$

where the $a_j \in \Re$ are coefficients to be determined. Differentiating (5) gives

$$y'(x) = p'(x) = \sum_{j=0}^{5} j a_j x^{j-1} \tag{6}$$

Evaluating (5) at the points $x_n, x_{n+\frac{1}{4}}, x_{n+\frac{1}{2}}, x_{n+\frac{3}{4}}$ and its first derivative (6) at the points $x_{n+\frac{3}{4}}, x_{n+1}$ leads to a system of six algebraic equations for the unknowns $a_0, a_1, a_2, a_3, a_4, a_5$ which when expressed in matrix form yields

$$\begin{pmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 \\ 1 & x_{n+\frac{1}{4}} & x^2_{n+\frac{1}{4}} & x^3_{n+\frac{1}{4}} & x^4_{n+\frac{1}{4}} & x^5_{n+\frac{1}{4}} \\ 1 & x_{n+\frac{1}{2}} & x^2_{n+\frac{1}{2}} & x^3_{n+\frac{1}{2}} & x^4_{n+\frac{1}{2}} & x^5_{n+\frac{1}{2}} \\ 1 & x_{n+\frac{3}{4}} & x^2_{n+\frac{3}{4}} & x^3_{n+\frac{3}{4}} & x^4_{n+\frac{3}{4}} & x^5_{n+\frac{3}{4}} \\ 0 & 1 & 2x_{n+\frac{3}{4}} & 3x^2_{n+\frac{3}{4}} & 4x^3_{n+\frac{3}{4}} & 5x^4_{n+\frac{3}{4}} \\ 0 & 1 & 2x_{n+1} & 3x^2_{n+1} & 4x^3_{n+1} & 5x^4_{n+1} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{pmatrix} = \begin{pmatrix} y_n \\ y_{n+\frac{1}{4}} \\ y_{n+\frac{1}{2}} \\ y_{n+\frac{3}{4}} \\ f_{n+\frac{3}{4}} \\ f_{n+1} \end{pmatrix} \tag{7}$$

Solving (7) for the unknowns where $t = x - x_n, 0 \le t \le 1$ gives the coefficients of the polynomial $p(t)$ in the form

$$p(t) = \alpha_0(t)y_n + \alpha_{\frac{1}{4}}(t)y_{n+\frac{1}{4}} + \alpha_{\frac{1}{2}}(t)y_{n+\frac{1}{2}} + \alpha_{\frac{3}{4}}(t)y_{n+\frac{3}{4}} + h\beta_{\frac{3}{4}}(t)f_{n+\frac{3}{4}} + h\beta_1(t)f_{n+1} \tag{8}$$

where

$$\alpha_0(t) = 1 - \frac{1064t}{111h} + \frac{11236t^2}{333h^2} - \frac{18544t^3}{333h^3} + \frac{14528t^4}{333h^4} - \frac{4352t^5}{333h^5}$$

$$\alpha_{\frac{1}{4}}(t) = \frac{864t}{37h} - \frac{4824t^2}{37h^2} + \frac{9840t^3}{37h^3} - \frac{8704t^4}{37h^4} + \frac{2816t^5}{37h^5}$$

$$\alpha_{\frac{1}{2}}(t) = -\frac{1224t}{37h} + \frac{9276t^2}{37h^2} - \frac{22672t^3}{37h^3} + \frac{22592t^4}{37h^4} - \frac{7936t^5}{37h^5}$$

$$\alpha_{\frac{3}{4}}(t) = \frac{2144t}{111h} - \frac{51304t^2}{333h^2} + \frac{134032t^3}{333h^3} - \frac{139520t^4}{333h^4} + \frac{50432t^5}{333h^5}$$

$$\beta_{\frac{3}{4}}(t) = -\frac{112t}{37} + \frac{2764t^2}{111h} - \frac{7576t^3}{111h^2} + \frac{8384t^4}{111h^3} - \frac{3200t^5}{111h^4}$$

$$\beta_1(t) = \frac{9t}{37} - \frac{78t^2}{37h} + \frac{232t^3}{37h^2} - \frac{288t^4}{37h^3} + \frac{128t^5}{37h^4}$$

Substituting these continuous coefficients into (8) and evaluating $p(t)$ at $t = 1$ and $p'(t)$ at $t = \frac{1}{4}, \frac{1}{2}, 0$ after a rearrangement gives the four discrete formulae

$$\left.\begin{aligned} y_{n+1} &= \frac{1}{37}y_n - \frac{8}{37}y_{n+\frac{1}{4}} + \frac{36}{37}y_{n+\frac{1}{2}} + \frac{8}{37}y_{n+\frac{3}{4}} + \frac{3h}{37}\left(4f_{n+\frac{3}{4}} + f_{n+1}\right) \\ y_{n+\frac{1}{4}} &= -\frac{19}{144}y_n + \frac{35}{16}y_{n+\frac{1}{2}} - \frac{19}{18}y_{n+\frac{3}{4}} + \frac{h}{192}\left(-37f_{n+\frac{1}{4}} + 29f_{n+\frac{3}{4}} + 2f_{n+1}\right) \\ y_{n+\frac{1}{2}} &= \frac{5}{153}y_n - \frac{13}{34}y_{n+\frac{1}{4}} + \frac{413}{306}y_{n+\frac{3}{4}} + \frac{h}{408}\left(-111f_{n+\frac{1}{2}} - 62f_{n+\frac{3}{4}} + 3f_{n+1}\right) \\ y_{n+\frac{3}{4}} &= \frac{133}{268}y_n - \frac{81}{67}y_{n+\frac{1}{4}} + \frac{459}{268}y_{n+\frac{1}{2}} + \frac{3h}{2144}\left(37f_n + 112f_{n+\frac{3}{4}} - 9f_{n+1}\right) \end{aligned}\right\} \quad (9)$$

Eq. (9) is the one-step embedded hybrid block method (EHBM) for solving eq. 1 numerically

Analysis of the New Block Method

Error constants and order

In this section, we consider the properties of the new block method (9) by studying the properties of (3) which include the local truncation error and order, consistency, and zero stability.

The local truncation error associated with eq. (3) is the linear operator L defined as

$$L[y(x);h] = \sum_{j=0}^{k} \alpha_{\frac{j}{k}} y\left(x + \tfrac{j}{k}h\right) - h\beta_{\frac{k-1}{k}} y'\left(x + \tfrac{k-1}{k}h\right) - h\beta_1 y'(x+h) \quad (10)$$

where $y(x)$ is an arbitrary function, continuously differentiable on $[a,b]$. Expanding $y\left(x+\tfrac{j}{k}h\right)$ and the derivatives $y'\left(x+\tfrac{k-1}{k}h\right)$ and $y'(x+h)$ as Taylor series about x, and collecting terms in (10) gives

$$L[y(x);h] = \sum_{j=0}^{k} c_j h^j y^{(j)}(x) = c_0 y(x) + c_1 h y'(x) + \ldots + c_q h^q y^{(q)}(x) + \ldots$$

where

$$\begin{aligned} c_0 &= \sum_{j=0}^{k} \alpha_{\frac{j}{k}} \\ c_1 &= \sum_{j=0}^{k} \left(\tfrac{j}{k}\alpha_{\frac{j}{k}} - \beta_{\frac{j}{k}}\right) \\ &\vdots \\ c_q &= \sum_{j=0}^{k} \left(\frac{\left(\frac{j}{k}\right)^q}{q!}\alpha_{\frac{j}{k}} - \frac{\left(\frac{j}{k}\right)^{q-1}}{(q-1)!}\beta_{\frac{j}{k}}\right), q = 2, 3, \ldots \end{aligned} \quad (11)$$

Thus, the method (3) is said to have order p if $c_0 = c_1 = \ldots = c_p = 0$ but $c_{p+1} \neq 0$ and c_{p+1} is called the error constant (see Lambert, 1973). Table 1 gives the order and error constant of the EHBM (9).

Table 1. Order and Error constant of EHBM (9)

Method	Order	Error constant
y_{n+1}	5	$-\dfrac{1}{378880}$
$y_{n+\frac{1}{4}}$	5	$\dfrac{41}{11796480}$
$y_{n+\frac{1}{2}}$	5	$-\dfrac{43}{25067520}$
$y_{n+\frac{3}{4}}$	5	$\dfrac{3}{548864}$

Following Lambert (1973), the EHBM (9) is consistent since its order $p=5>1$.

Stability Analysis

Applying the Dahlquist test equation $y'=\lambda y$ to (9), λ as a parameter and setting $z=h\lambda$ gives

$$\begin{bmatrix} 1-\frac{3}{37}z & \frac{8}{37} & \frac{-36}{37} & -\frac{8}{37}-\frac{12}{37}z \\ \frac{1}{96}z & 1+\frac{37}{192}z & \frac{-35}{16} & -\frac{29}{192}z+\frac{19}{18} \\ -\frac{1}{136}z & \frac{13}{34} & 1+\frac{37}{136}z & \frac{31}{204}z-\frac{413}{306} \\ \frac{27}{2144}z & \frac{81}{67} & -\frac{459}{268} & 1-\frac{21}{134}z \end{bmatrix}\begin{bmatrix} y_{n+1} \\ y_{n+\frac{1}{4}} \\ y_{n+\frac{1}{2}} \\ y_{n+\frac{3}{4}} \end{bmatrix}=\begin{bmatrix} 0 & 0 & 0 & \frac{1}{37} \\ 0 & 0 & 0 & -\frac{19}{144} \\ 0 & 0 & 0 & \frac{5}{153} \\ 0 & 0 & 0 & \frac{111}{2144}z+\frac{133}{268} \end{bmatrix}\begin{bmatrix} y_{n-\frac{3}{4}} \\ y_{n-\frac{1}{2}} \\ y_{n-\frac{1}{4}} \\ y_n \end{bmatrix} \quad (12)$$

Let $A=\begin{bmatrix} 1-\frac{3}{37}z & \frac{8}{37} & \frac{-36}{37} & -\frac{8}{37}-\frac{12}{37}z \\ \frac{1}{96}z & 1+\frac{37}{192}z & \frac{-35}{16} & -\frac{29}{192}z+\frac{19}{18} \\ -\frac{1}{136}z & \frac{13}{34} & 1+\frac{37}{136}z & \frac{31}{204}z-\frac{413}{306} \\ \frac{27}{2144}z & \frac{81}{67} & -\frac{459}{268} & 1-\frac{21}{134}z \end{bmatrix}$ and $B=\begin{bmatrix} 0 & 0 & 0 & \frac{1}{37} \\ 0 & 0 & 0 & -\frac{19}{144} \\ 0 & 0 & 0 & \frac{5}{153} \\ 0 & 0 & 0 & \frac{111}{2144}z+\frac{133}{268} \end{bmatrix}$ so that the stability polynomial of the method becomes

$$\rho(\xi,z)=\det(A\xi-B) \quad (13)$$

$$=\frac{20535}{18224}\xi^4-\frac{20535}{18224}\xi^3+\frac{4107}{18661376}\xi^3z^4-\frac{20535}{36488}\xi^4z+\frac{143745}{1166336}\xi^4z^2-\frac{34225}{2332672}\xi^4z^3$$
$$+\frac{4107}{4665344}\xi^4z^4-\frac{20535}{72896}\xi^3z-\frac{20535}{116636}\xi^3z^2+\frac{6845}{4665344}\xi^3z^3$$

Zero stability and convergence

The EHBM (9) is said to be zero stable if no roots of the first characteristic polynomial $\rho(\xi, z)$ has modulus greater than one and if every root of modulus one is simple. Thus, setting $z = 0$ in (13) yields $\rho(\xi;0) = \frac{20535}{18224}\xi^3(\xi - 1) = 0$. Hence, $|\xi_1| = 1, |\xi_{2,3,4}| = 0$ which confirms the zero - stability of the EHBM (9) (see Fatunla, 1991). Accordingly, since consistency and zero-stability of the EHBM (9) have been established, the method is indeed convergent (see Lambert, 1973).

Absolute stability

To obtain the stability region of the EHBM (9), the stability polynomial (13) and its derivative are used to plot the region of absolute stability as indicated in figure 1.

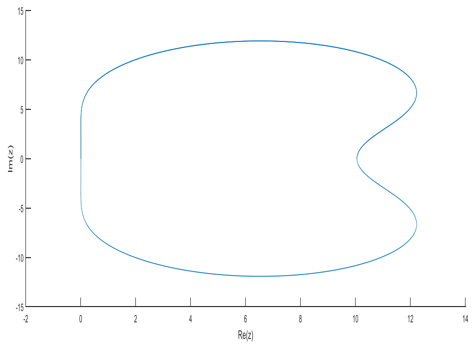

Figure 1: Region of Absolute stability of EHBM (9).

The following definition will aid the interpretation of figure 1.

A numerical method is said to be A-stable if its region of absolute stability contains the whole of the left-hand plane $(\mathrm{Re}\,\lambda h < 0)$ (Dahlquist, 1963)

Figure 1 clearly shows that the EHBM (9) is *A*-stable.

Numerical Experiments

We compare results of the EHBM (9) with some numerical problems in the literature as follows:

EBDF = Method by Ezzeddine and Hojjati (2011) for $k = 4$.

HEBDF = Another method in Ezzeddine and Hojjati (2011) for $k = 4$.

ECBBDF = Method by Akinfenwa and Jator (2015) for $k = 4, 6$.

EHBM = Method (9) for $k = 1$.

Example 1: We consider a stiff system solved by Ezzeddine and Hojjati (2011) and Akinfenwa and Jator (2015).

$$y_1' = -y_1 - 30y_2 + 30e^{-x} \qquad y_1(0) = 1$$

$$y_2' = 30y_1 - y_2 - 30e^{-x} \qquad y_2(0) = 1, \quad 0 \le x \le 20, \quad h = 0.01$$

The stiffness ratio of the system is 1:200 and its exact solution is $y_1(x) = e^{-x}, y_2(x) = e^{-x}$. The results are shown in Table 2.

Table 2: Absolute Errors for Example 1 for $h = 0.01$

x	y_i	Error in EBDF $k=4, p=5$	Error in HEBDF $k=4, p=5$	Error in ECBBDF $k=4$	Error in EHBM $k=1$
1.0	y_1	1.71×10^{-13}	8.15×10^{-15}	1.28×10^{-15}	1.11×10^{-15}
	y_2	2.60×10^{-12}	8.48×10^{-13}	1.17×10^{-14}	8.88×10^{-16}
10.0	y_1	5.03×10^{-17}	9.83×10^{-18}	1.08×10^{-19}	1.22×10^{-19}
	y_2	3.36×10^{-16}	7.71×10^{-17}	1.62×10^{-18}	3.86×10^{-19}
20.0	y_1	1.17×10^{-20}	1.29×10^{-21}	7.24×10^{-23}	5.33×10^{-23}
	y_2	7.83×10^{-21}	2.79×10^{-21}	5.29×10^{-23}	2.77×10^{-23}

Example 2: We consider another stiff system of initial value problem solve by Akinfenwa and Jator (2015) for different step sizes:

$$y'(x) = \begin{pmatrix} -21 & 19 & -20 \\ 19 & -21 & 20 \\ 40 & -40 & -40 \end{pmatrix} y(x), \quad y(0) = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \qquad 0 \le x \le 20$$

The three components of the theoretical solution of the problem are given as

$$y(x) = \tfrac{1}{2}\begin{pmatrix} e^{-2x} + e^{-40x}(\cos(40x) + \sin(40x)) \\ e^{-2x} - e^{-40x}(\cos(40x) + \sin(40x)) \\ 2e^{-40x}(\sin(40x) - \cos(40x)) \end{pmatrix}$$

The main aim is to show the order and accuracy of the EHBM for different choices of the constant step size *h*. The results are shown in Table 3.

Table 3: Maximum errors for Example 2

h	$k=4$ ECBBDF Order $m=5$	$k=6$ ECBBDF Order $m=5$	$k=1$ EHBM Order $m=5$
0.01	3.08×10^{-4}	9.88×10^{-5}	2.52×10^{-8}
0.005	7.77×10^{-6}	1.76×10^{-6}	2.54×10^{-10}
0.0025	1.41×10^{-7}	2.69×10^{-8}	6.74×10^{-12}
0.00125	2.31×10^{-9}	3.96×10^{-10}	1.07×10^{-13}
0.00625	6.26×10^{-12}	6.26×10^{-12}	1.61×10^{-14}

DISCUSSION AND CONCLUSION

We derived an A-stable one-step embedded hybrid block method for the solution of stiff first-order ordinary differential equations using multistep collocation approach. The method developed was also found to be zero-stable and consistent and as such convergent. The new embedded hybrid block method though has a lower step number $k=1$ yielded the least absolute errors when compared with higher step numbers $k=4,6$ block methods on the same problems solved by previous researchers. Clearly, the results from the three examples indicate that the new method competes favourablly with other existing methods.

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