A Collocation Method Based on Euler and Bernoulli Polynomials for the Solution of Volterra Integro-Differential Equations

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ABSTRACT

In this research, we constructed collocation methods for approximating the solutions of Volterra integro-differential equations using Bernoulli polynomials and Euler polynomials as basic functions. Sample problems ranging from linear first to second order Volterra integro-differential (VIDEs) equations using the methods developed were solved. The method was implemented using MAPLE 17 and MATLAB software and the obtained results are compared with the exact solution for the polynomials. Results revealed that Bernoulli Polynomials has the best accuracy for the first order and second order VIDE. However, both polynomials offer good approximations.

Keywords: Collocation, Euler, Bernoulli, Polynomials.

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INTRODUCTION

The method developed here is the extension of the work of Adesanya, A. O., Yahaya, Y. A., Ahmed, B. and Onsachi, R. O. (2019) from Boubakar polynomials to other polynomials. The integro-differential equation is converted into an integral equation and transformed into a system of linear equations. The methods of solution for initial value problems for integro-differential equations of the Volterra type has been considered by several authors in the past. Amongst them are Awoyemi and Kayode (2005), Adesanya et al. (2009). These papers independently implemented their methods in predictor-corrector mode which is believed to have some setbacks. In order to cater for the shortcoming of the predictor-corrector methods, the block method was adopted. The block method gives solutions at each grid within the interval of integration without overlapping, and the burden of developing separate predictors is eradicated.

Bernoulli and Euler polynomials have many useful properties. However, the main disadvantage of these polynomials is that they are not orthogonal but orthonormal.

Mirzaee and Samadyar (2020) offered an explicit representation of orthonormal Bernoulli polynomials where they showed that these polynomials can be created from a linear combination of standard basis polynomials. The Bernoulli polynomials play an important role in several branches of mathematical analysis, such as the theory of distributions in p-Adic analysis (Koblitz, 1984), the theory of modular forms (Lang, 1976), the study of polynomial expansions of analytic functions (Boas and Buck, 1964), etc. Recently, new applications of the Bernoulli polynomials have also been found in mathematical physics, in connection with the theory of the Korteweg-de Vries equation (Fairlie and Veselov, 2001), Lamé equation (Grosset and Veselov, 2006), and in the study of vertex algebras (Doyon, B., Lepowsky, J. and Milas, A. (2006)). Many authors also used Bernoulli polynomials in different ways to find numerical solution of many complex problems, such as numerical approximation for generalized pantograph equation using Bernoulli matrix method (Tohidi, E., Bhrawy, A. H. and Erfani, K. (2013)), numerical solution of second-order linear system of partial differential equations using Bernoulli polynomials (Tohidi and Kilicman, 2013), numerical solution of Volterra type integral equations by means of Bernoulli polynomials (Mohsenyzadeh, 2016). Udaya (2020) proposed a method to find polynomial approximation to solution of linear integro-differential equations (IDEs) by application of an operational matrix developed from a class of modified Bernoulli polynomials.
Recently, a Euler wavelet numerical method has been presented to solve the nonlinear Volterra integro-differential equations where the Euler wavelets are constructed by Euler polynomials (Wang and Zhu, 2017). Because of their less term, Euler polynomials are believed to have many advantages over other polynomials in approximating arbitrary functions.

Bernoulli polynomials

The Bernoulli polynomials are a generalization of the Bernoulli numbers. They have a variety of interesting properties.
The Bernoulli polynomials are a sequence of polynomials, $B_n(x)$, defined by means of the following exponential generating functions:

$$\frac{te^{xt}}{e^{xt}-1}=\sum_{n=0}^{\infty}\frac{t^n}{n!}B_n(x) \quad (1)$$

The generating function for the Bernoulli polynomials is the generating function for the Bernoulli numbers multiplied by a term of e^{xy}. e^{xy} There are further connections: we can use the generating function for the Bernoulli numbers to develop a recurrence relation for the Bernoulli polynomials.
With the generating function of Bernoulli numbers B_n of the first order are shown as below

$$\frac{t}{e^{xt}-1}=\sum_{n=0}^{\infty}\frac{t^n}{n!}B_n. \quad (2)$$

Definition 1 (Bernoulli Polynomials) (Takao *et al*, 2016)
The function $e^{xt+g(t)}$ $e^{xt+g(t)}$ generates the β polynomials of order zero, where if
we put $g(t) = 0$ we will obtain the simplest polynomials of this kind e^{xt}. e^{xt} These β polynomials are known as Bernoulli polynomials of order zero which are defined
as follows :

$$B_n^{(0)}(x) = x^n \quad (3)$$

Therefore, we have

$$e^{xt}=\sum_{n=0}^{\infty}\frac{t}{n!}B_n^0(x)$$

$$=\sum_{n=0}^{\infty}\frac{(xt)^n}{n!}=\sum_{n=0}^{\infty}\frac{t^n}{n!}B_n^0(x). \quad (4)$$

Using (1), we can expand this definition to Bernoulli polynomials of order α given by the identity

$$\frac{t^\alpha e^{xt}}{(e^t-1)^\alpha}=\sum_{n=0}^{\infty}\frac{t^n}{n!}B_n^{(\alpha)}(x) \quad (5)$$

Putting $x = 0$, we will obtain

$$\frac{t^\alpha}{(e^t-1)^\alpha}=\sum_{n=0}^{\infty}\frac{t^n}{n!}B_n^{(\alpha)} \quad (6)$$

Considering $\alpha = 1$ that is the order of the polynomial, the generating function of the Bernoulli polynomial becomes

$$\frac{t}{(e^t-1)} = \sum_{n=0}^{\infty} \frac{t^n}{n!} B_n. \qquad (7)$$

Let $B_n(x)$ be the Bernoulli polynomial of order 1then,

$$\frac{d}{dx}(B_n(x)) = nB_{n-1}(x) \qquad (8)$$

and

$$\int_a^x B_n(t)dt = \frac{1}{n+1}\big(B_{n+1}(x) - B_{n+1}(a)\big). \quad (9)$$

For any integer $n \geq 0$; we have

$$B_n(x) = (B+x)^n \qquad (10)$$

Where $B^{n\prime}s$ are Bernoulli numbers and we have the first few Bernoulli polynomials as;

$$\left.\begin{aligned} B_0(x) &= 1 \\ B_1(x) &= x - \tfrac{1}{2} \\ B_2(x) &= x^2 - x + \tfrac{1}{6} \\ B_3(x) &= x^3 - \tfrac{3}{2}x^2 + \tfrac{1}{2}x \end{aligned}\right\} \qquad (11)$$

Euler polynomials

The Euler polynomials $E_n(x)$ of order α are usually defined by means of the following exponential generating functions:

$$\frac{2^\alpha e^{xt}}{e^t+1} = \sum_{n=0}^{\infty} E_n^\alpha(x)\frac{t^n}{n!}, \quad |t| < \pi \qquad (12)$$

With $E_n^\alpha(0)$ being the Euler numbers which can be expressed as

$$E_n^\alpha(0) = 2^{-n}C_n^{(\alpha)} \qquad (13)$$

For Euler polynomials of order one, $\alpha = 1$ and the generating function becomes

$$\frac{2e^{xt}}{e^t+1} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}, \qquad (14$$

Euler polynomials are also referred to as η polynomials (Srivastava, and Pinter 2004) therefore

$$E_n^{(\alpha)} = \left(\frac{1}{2}C^{(\alpha)} + x\right)^n \qquad (15)$$

$$\frac{d}{dx}E_n^{(\alpha)}(x) = nE_{n-1}^{(\alpha)}(x) \qquad (16)$$

The generating function is given as

$$E_n(x) = x^n - \frac{1}{2}\sum_{i=0}^{n-1}\binom{n}{i} E_i(x) \qquad (17)$$

And the first few Euler polynomials are given below as:

$$\left.\begin{aligned} E_0(x) &= 1 \\ E_1(x) &= x - \frac{1}{2} \\ E_2(x) &= x^2 - x \\ E_3(x) &= \left(x^3 - \frac{3}{2}x^2 + \frac{1}{4}\right) \\ E_4(x) &= (x^4 - 2x^3 + x) \end{aligned}\right\} \qquad (18)$$

We consider the Volterra integro-differential equation

$$y'(x) = f(x) + \lambda \int_a^x K(x,t)y(t)dt, \qquad (19)$$

with the initial condition

$$y(a) = y_0 \qquad (20)$$

on $a \le x \le b$
The approximate solution is given in the form

$$y_N(x) = P(x)A \qquad (21)$$

where $\boldsymbol{P}(x) = [P_0(x), P_1(x), \cdots, P_N(x)]$ and $A = [a_0, a_1, \cdots, a_N]^T$.

The integro-differential equation (19) is converted to an integral equation by integrating on $[a, x]$ so that (19) can be written as

$$\int_a^x y'(\xi)d\xi = \int_a^x f(\xi)d\xi + \lambda \int_a^x \left[\int_a^x K(x,t)y(t)dt\right] dx,$$

so that

$$y(x) = y_0 + \int_a^x f(\xi)d\xi + \lambda \int_a^x \left[\int_a^x K(x,t)y(t)dt\right] dx. \qquad (22)$$

Substituting (21) into (22) yields

$$P(x)A = y_0 + \int_a^x f(\xi)d\xi + \lambda \int_a^x \left[\int_a^x K(x,t)P(t)Adt\right] dx,$$

$$\left[P(x) - \lambda \int_a^x \left[\int_a^x K(x,t)P(t)Adt\right] dx\right] A = y_0 + \int_a^x f(\xi)d\xi \qquad (23)$$

which can be written as

$$\mu(x)A = \tau(x) \qquad (24)$$

where

$$\mu(x) = P(x) - \lambda \int_a^x \left[\int_a^x K(x,t)P(t)Adt\right] dx \quad (25)$$

Where $\boldsymbol{P}(x)$s are the Bernoulli and Euler polynomials respectively 5 and

$$\tau(x) = y_0 + \int_a^x f(\xi)d\xi \qquad (26)$$

We will collocate (24) at $N+1$ points to give

$$\mu(x_i)A = \tau(x_i) \qquad (27)$$

We will solve (27) to get the coefficients$a_0, a_1, \cdots, a_N$.

Numerical examples

Application of the collocation method based on Bernoulli and Euler polynomials

The selected examples will be evaluated using MAPLE 17 and MATHLAB Softwares respectively. The obtained results will be compared for the two polynomials namely Bernoulli and Euler polynomials.

The work of Adesinya *et al*., (2019) considered only integral equations and only first order integro-differential equations using Boubakar polynomials while we extended to include second order integro-differential equations using Bernoulli and Euler polynomials, hence the choice of our examples.

Example 1

Let us consider the Volterra integro-differential equation of the first order in Day (1967) given by

$$\left.\begin{array}{c} y'(x) + y(x) = (x^2 + 2x + 1)e^{-x} \\ +5x^2 + 8 - \int_0^x ty(t)dt\,, \\ 0 \le x \le 1 \\ y(0) = 10 \end{array}\right\} \qquad (28)$$

The exact solution is $y(x) = 10 - xe^{-x}$.

Let

$$y_N(x) = B(x)A \qquad (29)$$

Where $B(x)$ are Bernoulli polynomials. Substituting (29) in to (28) and simplifying we have

$$\left[B(x) - \lambda \int_a^x \left[\int_a^x K(x,t)B(t)dt\right]dx\right]A = y_0 + \int_a^x f(\xi)d\xi \qquad (30)$$

where $B(x) = [B_0(x), B_1(x), \cdots, B_N(x)]$ and $A = [a_0, a_1, \cdots, a_N]^T$,

$K(x,t) = t$, $y_0 = 10$ and $f(\xi) = (\xi^2 + 2\xi + 1)e^{-\xi} + 5\xi^2 + 8$

Substituting and solving for $a_j{}'s$, we have the solutions is

$$\begin{aligned} y_B(x) = 10.00 &- 1.000000894\,x + 1.000012577\,x^2 - 0.5000913917\,x^3 + 0.1670639779\,x^4 \\ &- 0.04276208036\,x^5 + 0.01029381129\,x^6 - 0.003657544304\,x^7 \\ &+ 0.001834744372\,x^8 - 0.0006929021646\,x^9 + 0.0001202523057\,x^{10} \end{aligned} \qquad (31)$$

$y_E(x) =$
$10.00000000 - 1.000000895\,x + 1.000012608\,x^2 - 0.5000916617\,x^3 + 0.1670653461\,x^4 - 0.04276640692\,x^5 + 0.01030251176\,x^6 - 0.003668629784\,x^7 + 0.001843388610\,x^8 - 0.0006966589010\,x^9 + 0.0001209483997\,x^{10}$ (32)

Example 2

Let us consider the Volterra integro-differential equation of the second order in Wazwaz (2011) given by

$$\left.\begin{array}{c} y''(x) = (1+x) + \int_0^x (x-t)y(t)dt, \\ 0 \le x \le 1 \\ y(0) = 1 \\ y'(0) = 1 \end{array}\right\} \quad (33)$$

The exact solution is $y(x) = e^x$.

$$y_N(x) = B(x)A \quad (34)$$

Where $\boldsymbol{B}(x)$ are Bernoulli polynomials. Substituting (34) into (33) and simplifying we have

$$\left[B(x) - \lambda \int_0^x \left[\int_o^x \left[\int_0^x K(x,t)B(t)dt\right] dx\right]\right] A = y_0{}' + y_0 + \int_0^x \left(\int_0^x f(\xi)d\xi\right) d\xi \quad (35)$$

Where $\boldsymbol{B}(x) = [B_0(x), B_1(x), \cdots, B_N(x)]$ and $\boldsymbol{A} = [a_0, a_1, \cdots, a_N]^T$,
$K(x,t) = x - t$, $y_0 = 1$, $y_0{}' = 1$ and $f(\xi) = (1+\xi)$

With the help of Maple computer software, we Solve for the $a_j{}'s$ and the solutions are respectively

$y_B(x) =$
$0.9999999998 + 0.9999999996x + 0.4999999851x^2 + 0.1666667843x^3 + 0.04166617519x^4 + 8.334502761 \times 10^{-3}x^5 + 1.387351362 \times 10^{-3}x^6 + 1.992855588 \times 10^{-4}x^7 + 2.505166916 \times 10^{-5}x^8 + 2.148813331 \times 10^{-6}x^9 + 5.431861356 \times 10^{-7}x^{10}$ (36)

$y_E(x) = 0.9999999997 + 4.999999877 \times 10^{-1}x^2 + 1.666667738 \times 10^{-1}x^3 + 4.166613439 \times 10^{-2}x^4 + 1.614947 \times 10^{-6}x^5 + 1.385812884 \times 10^{-3}x^6 + 0.2020725744 \times 10^{-4}x^7 + 2.221468311 \times 10^{-5}x^8 + 9.305392300 \times 10^{-7}x^9 + 1.970817754 \times 10^{-7}x^{10}$ (37)

DISCUSSION

Integro-differential equations are difficult to solve analytically, therefore it is necessary to obtain the approximate solutions for Integro-differential equations. For this purpose, the collocation method based on Euler Polynomials and Bernoulli polynomials are developed in order to find the approximate solutions of Volterra integro-differential equations. The approximate solutions are obtained as a convergent polynomial series. A considerable advantage of this method is that the coefficients of the basis polynomials of the solution can be obtained by using the computer programs.

The solution by the method based on Bernoulli polynomials when compared with the exact solution, it gives a maximum absolute error of 2.4×10^{-8}. while the method with Euler polynomials gives the maximum absolute error at selected point of 2.5×10^{-8}.

Table 1 Solution to Example 1 with Bernoulli and Euler **Polynomial and Absolute Error**

x	**Exact Solution**	**Solution with Bernoulli Polynomials**	**Absolute Error**	**Solution with Euler polynomials**	**Absolute Error**
0.0	10	10	0	10	0
0.1	9.909516258	9.909516234	2.4×10^{-8}	9.909516	2.5×10^{-8}
0.2	9.836253849	9.836253828	2.1×10^{-8}	9.836254	2.0×10^{-8}
0.3	9.777754534	9.777754513	2.1×10^{-8}	9.777755	1.8×10^{-8}
0.4	9.731871982	9.731871962	2×10^{-8}	9.731872	1.8×10^{-8}
0.5	9.69673467	9.696734654	1.6×10^{-8}	9.696735	1.6×10^{-8}
0.6	9.670713018	9.670713005	1.3×10^{-8}	9.670713	1.4×10^{-8}
0.7	9.652390287	9.652390275	1.2×10^{-8}	9.65239	1.1×10^{-8}
0.8	9.640536829	9.640536817	1.2×10^{-8}	9.640537	1.0×10^{-8}
0.9	9.634087306	9.634087296	1.0×10^{-8}	9.634087	1.0×10^{-8}
1.0	9.632120559	9.632120547	1.2×10^{-8}	9.632121	9.0×10^{-9}

It is obvious that the solutions of the problem in example 1 by using the two polynomials, that is Bernoulli and Euler polynomials reveals that collocation method with all the two orthogonal polynomials shows a similar result, however, Bernoulli polynomials has the least absolute error and can be adjudged the most accurate method since its absolute error is the smallest as compared to that of Euler polynomials.

Table 2 Solution to Example 2 with Bernoulli and Euler Polynomial and Absolute Error

x	**Exact Solution**	**Solution with Bernoulli Polynomials**	**Absolute Error**	**Solution with Euler Polynomials**	**Absolute Error**
0.00	1	0.9999999998	2×10^{-10}	1.0	0
0.1	1.105170918	1.105170918	0	1.105170918	0
0.2	1.221402758	1.221402758	0	1.221402759	1.0×10^{-9}
0.3	1.349858808	1.349858808	0	1.349858807	1.0×10^{-9}
0.4	1.491824698	1.491824698	0	1.491824698	0
0.5	1.648721271	1.64872127	1×10^{-9}	1.64872127	1.0×10^{-9}
0.6	1.8221188	1.8221188	0	1.8221188	0
0.7	2.013752707	2.013752708	1×10^{-9}	2.013752708	1.0×10^{-9}
0.8	2.225540928	2.225540928	0	2.225540928	0
0.9	2.459603111	2.45960311	1×10^{-9}	2.459603111	0
1.0	2.718281828	2.718281827	1×10^{-9}	2.718281826	2.0×10^{-9}

Example 2 is a second order Volterra integro-differential equation curled from Wazwaz (2011) with the exact solution given as $y(x) = e^x$. We test the method on a second order Volterra Integro-Differential Equation and the compared values of the results of the proposed method with exact solution of Example 2 using Bernoulli polynomials as basic functions, gives a maximum absolute error of 1.0×10^{-9} whereas, the method with Euler polynomial compared with the exact solution gives a maximum absolute error of 2.0×10^{-9}.

The method with Bernoulli polynomials gives a more accurate result as compared to that with Euler polynomials. The coefficients $a_j's$, which determines the approximate solutions were obtained using MATLAB and MAPLE 17 computer software.

Recommendations

Based on the results obtained in this work as tested with the simplest form of Volterra integro-differential equations, we recommend the application of this approach for solving more complicated problems modelled in Volterra integro-differential equations arising in both physical and engineering sciences. Some of the areas where similar problems could be found include fluid dynamics, biological models such as population dynamics, chemical kinetics and economics.

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