A Collocation Method for the Solution of Volterra Integro-differential Equations Based on Orthogonal Polynomials

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ABSTRACT

In this research, we constructed collocation methods for approximating the solutions of Volterra integro-differential equations using Legendre polyomials, Chebyshev polynomials and Laguerre polynomials as basis functions. The Standard Collocation Method (SCM) is used to determine the desired collocation points within an interval say, to solve sample problems ranging from linear first to second order Volterra integro-differential (VIDEs) equations using the methods.developed The method was implemented using MAPLE 17 computer software and the obtained results are compared with the exact solution for the orthogonal polynomials. Results revealed that Laguerre Polynomials has the best accuracy for the first order VIDE while Legendre polynomial has the best approximation for the second order VIDE. However both polynomials offer good approximations.

INTRODUCTION

For a number of decades, functional equations have had a significant attention in mathematics. Recently, attention has been paid to functional equations with an integral as they offer powerful techniques for modeling a variety of physical problems. According to Bo, T. L. Xie, L. and Zheng, X. J. (2007), Integro-differential equations have been found to describe various kinds of phenomena, such as glass forming process, drop wise condensation, nano-hydrodynamics, wind ripple in the desert and many other systems in science. They also have wide applications in fluid dynamics, physics, chemistry, astronomy, biology, epidemiology, for instance, when the models are age-structured or spatial epidemics, and economics. Orthogonal polynomials occur often as solutions of mathematical and physical problems. They play an important role in several fields of study including wave mechanics, heat conduction, electromagnetic theory, quantum mechanics, medicine and mathematical statistics (Pipe and Zwart, 2014). They provide a natural way to solve, expand, and interpret solutions to many types of important equations.

Literature Review

It is usually difficult and sometimes impossible to obtain exact solutions of integro-differential equation. Most of the existing methods developed are discrete methods which do not offer good accuracy for example Linz (1969), Day (1967), Al-Jubury (2010), Filiz (2013) etc. The field of orthogonal polynomials is an active research area in mathematics as well as with applications in mathematical physics, engineering, and computer science. Representation of a smooth function in terms of a series expansion using orthogonal polynomials is a fundamental concept in approximation theory and forms the basis of spectral methods of solution of Volterra integro-differential equations with functional arguments (Burcu, G., Mehmet, S. and CoGkun, G., 2014). Legendre polynomials have been used extensively in the solution of the boundary value problems and in computational fluid dynamics (Elbarbary, 2005). The solution of mixed integral equation (MIE) of the first and second kind in time and position is discussed and obtained in the space

$$L^2\,[-1,1] \times C[0,T], T < 1$$

by (Abdou and Elsayed, 2015), they in addition, solved the FIE of the second kind, with singular kernel, using Legendre polynomials. Chebyshev polynomials are well-known family of orthogonal polynomials on the interval [-1,1] that have many applications (Khader, 2012). They are widely used because of their good properties in the approximation of functions. However, with our best knowledge, very little work was done to adapt these polynomials to the solution of integro-differential equations. Chebyshev polynomials have a very concise form making it of leading importance among orthogonal polynomials. Jabari, D. S., Ezzati, R. and Maleknejad, K., (2017), considered a system of dual integral equations with trigonometric kernels and converted them to Cauchy-type singular integral equations. They used the Chebyshev orthogonal polynomials to construct approximate solution for Cauchy-type singular integral equations which they used to solve the main dual integral equations. Burcu et al., 2014), applied the Laguerre collocation method for solving a class of Fredholm integro-differential equations with functional arguments. Yüzbaşı (2014), presented a collocation method based on Laguerre polynomials to solve the pantograph-type Volterra integro-differential equations under the initial conditions. Baykuş and Sezer (2016), developed a Laguerre matrix method to find an approximate solution of linear differential, integral and integro-differential equations with variable coefficients under mixed conditions in terms of Laguerre polynomials. Dilek and Ayşegül (2018), presented an approximation method based on the Laguerre polynomials for fractional linear Volterra integro-differential equations.

The need for the comparison of the methods based on Laguerre, Legendre and the Chebyshev polynomials to find the solution of Volterra integro-differential equations led us to develop an effective and accurate numerical methods for the solution of linear and nonlinear integro-differential equations of varying orders.

METHODS

Our starting point is the generation of orthogonal polynomials by a three-term recurrence relation, which leads to some very useful formulas.

Definition 1 Orthogonal Polynomials (Shen *et al.*, 2011)

Given an open interval $I = (a, b)\ (-\infty \leq a < b \leq +\infty)$ and a generic weight function $\omega(x)$

such that $\omega(x) > 0$ for all $x \in I$

, and $\omega(x) \in L^1(I)$ (1)

two functions $f(x)$ and $g(x)$ are said to be orthogonal to each other in $L^2_{\omega(x)}(a,b)$ or orthogonal with respec $\omega(x)$ if

$$(f,g)_\omega = \int_a^b f(x)g(x)\omega(x)dx = 0 \quad (2)$$

It follows that, if an algebraic polynomial of degree n

denoted by

$$P_n(x) = k_n x^n + k_{n-1}x^{n-1} + \cdots + k_1 x + k_0, k_n \neq 0 \quad (3)$$

where $\{k_i\}$ are real constants, and k_n is the leading coefficient of P_n

. Then a sequence of polynomials $\{P_n\}_{n=0}^{\infty}$ with $deg(P_n) = n$ is said to be orthogonal in $L^2_\omega(a,b)$ if

$$(P_n, P_m)_\omega = \int_a^b P_n(x)P_m(x)\omega(x)dx = \gamma_n \delta_{nm} \quad (4)$$

where the constant $\gamma_n = \|P_n\|^2_\omega$ is nonzero, and δ_{nm} is the Kronecker delta.

Derivation of a Collocation Method based on Orthogonal Polynomials

We consider the numerical solution to the general nth order initial value problem of the Voltera type integro-differential equations of the form

$$\left.\begin{aligned} y^{(n)}(x) &= F(x) + \int_0^x K(x,t,y(t))dt \\ y(x_0) &= y_0, \ldots, y^{(n-1)}(x_0) = y_0^{(n-1)} \end{aligned}\right\} \quad (5)$$

which can be written in linear form as

$$y^{(n)}(x) = F(x) + \int_0^x K(x,t)y(t)dt \quad (6)$$

Equation (5) can be written as an initial value problem as follows:

$$\left.\begin{aligned} y^{(n)}(x) &= F(x) + \int_0^x K(x,t)y(t)dt \\ y(x_0) &= y_0, \\ y'(x_0) &= y_0', \ldots, y^{(n-1)}(x_0) = y_0^{(n-1)} \end{aligned}\right\} \quad (7)$$

By implication, $y^{(j)} = \frac{d^j}{dx^j}y(x)$

, and $y^{(0)} = y$ The method we are describing is an extension of the work of Olumuyiwa, A., Agbolade, I., and Anake T. A. (2017).

We seek the unknown function $y(x)$ given K Let

$$y_N(x) = \sum_{j=0}^{N} a_j P_j(x) \quad (8)$$

Where $P_j(x)$
are orthogonal polynomials (Legendre polynomials, Chebyshev polynomials and Laguerre polynomials)
Then,

$$y_N^{(n)}(x) = \sum_{j=0}^{N} a_j P_j^{(n)}(x) \quad (9)$$

For any arbitrary choice of N
, and considering an order ordinary differential equation, we substitute the approximate solution (8) into the equation (6) to have

$$\sum_{j=0}^{N} a_j P_j^{(n)}(x) = F(x) + \int_0^x K(x,t) \sum_{j=0}^{N} a_j P_j(t)dt \quad (10)$$

That is,

$$\sum_{j=0}^{N} a_j \left(P_j^{(n)}(x) - \int_0^x K(x,t)P_j(t)dt \right) = F(x) \quad (11)$$

Simplifying we have

$$a_0\left(P_0^{(n)}(x) - \int_0^x K(x,t)P_0(t)dt\right) + a_1\left(P_1^{(n)}(x) - \int_0^x K(x,t)P_1(t)dt\right) + a_2\left(P_2^{(n)}(x) - \int_0^x K(x,t)P_2(t)dt\right) + a_3\left(P_3^{(n)}(x) - \int_0^x K(x,t)P_3(t)dt\right) + \cdots + a_N\left(P_N^{(n)}(x) - \int_0^x K(x,t)P_N(t)dt\right) = F(x) \quad (12)$$

Where $F(x)$ and $K(x,t)$ are known functions. Equation (12) can be expressed as

$$F(x) = a_0\sigma_0(x) + a_1\sigma_1(x) + a_2\sigma_2(x) + \cdots + a_{N-1}\sigma_{N-1}(x) + a_N\sigma_N(x) \quad (13)$$

where

$$\sigma_j = P_j^{(n)}(x) - \int_0^x K(x,t)P_j(t)dt \quad (14)$$

We have $N+1$ ınknowns. The first n unknowns can be obtained from the initial conditions. The remaining equations are to be determined by collocation method using the $N+1-n$

collocation points $(x_1, x_2, \dots, x_{N+1-n})$
.Now, by equation (8)

$$y_N(x) = a_0P_0(x) + a_1P_1(x) + a_2P_2(x) + \cdots + a_NP_N(x) \quad (15)$$

Then,

$$\left.\begin{aligned} y_N(x_0) &= a_0P_0(x_0) + a_1P_1(x_0) + \cdots + a_NP_N(x_0) = y_0 \\ y_N'(x_0) &= a_0P_0'(x_0) + a_1P_1'(x_0) + \cdots + a_NP_N'(x_0) = y_0' \\ &\vdots \\ y_N^{(n-1)}(x_0) &= a_0P_0^{(n-1)}(x_0) + a_1P_1^{(n-1)}(x_0) + \cdots + a_NP_N^{(n-1)}(x_0) = y_0^{(n-1)} \end{aligned}\right\} \quad (16)$$

and by equation (13) At the determined collocation points x_n

,

$$n = 1,2, \ldots N + 1 - n$$

, we arrive at the following

$$\left.\begin{aligned} F(x_1) &= a_0\sigma_0(x_1) + a_1\sigma_1(x_1) \\ &+ \cdots + a_N\sigma_N(x_1) \\ F(x_2) &= a_0\sigma_0(x_2) + a_1\sigma_1(x_2) \\ &+ \cdots + a_N\sigma_N(x_2) \\ &\vdots \\ F(x_{N+1-n}) &= a_0\sigma_0(x_{N+1-n}) + a_1\sigma_1(x_{N+1-n}) \\ &+ \cdots + a_N\sigma_N(x_{N+1-n}) \end{aligned}\right\}$$

Combining (16) and (17) gives the following matrix

$$\begin{pmatrix} P_0(x_0) & P_1(x_0) & \cdots & P_N(x_0) \\ P_0'(x_0) & P_1'(x_0) & \cdots & P_N'(x_0) \\ \vdots & \vdots & \vdots & \vdots \\ P_0^{(n-1)}(x_0) & P_1^{(n-1)}(x_0) & \cdots & P_N^{(n-1)}(x_0) \\ \sigma_0(x_1) & \sigma_1(x_1) & \cdots & \sigma_N(x_1) \\ \sigma_0(x_2) & \sigma_1(x_2) & \cdots & \sigma_N(x_2) \\ \vdots & \vdots & \vdots & \vdots \\ \sigma_0(x_{N+1-n}) & \sigma_1(x_{N+1-n}) & \cdots & \sigma_N(x_{N+1-n}) \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ \vdots \\ a_N \end{pmatrix} = \begin{pmatrix} y_0 \\ y_0' \\ \vdots \\ y_0^{(n-1)} \\ F(x_1) \\ F(x_2) \\ \vdots \\ F(x_{N+1-n}) \end{pmatrix} \quad (18)$$

which can be used by any convenient method of solving algebraic equations to obtain the unknown $a_j's$

Numerical Examples

Standard Collocation Method (SCM)

The standard collocation method (SCM) is used to determine the desired collocation points within an interval say, $[\vartheta, \sigma]$

, to solve sample problems ranging from linear to nonlinear first and second order Volterra Integro-Differential Equations (VIDEs) using the method described above and is given by

$$x_p = \vartheta + \frac{(\sigma - \vartheta)}{N}p, \qquad p = 1,2,3, \ldots N \qquad (19)$$

The selected examples will be evaluated using MAPLE 17 and MATHLAB. The obtained results will be compared for three orthogonal polynomials namely Legendre polynomials, Laguerre polynomials and the Chebyshev polynomials.

Example 1

Let us consider the Volterra integro-differential equation of the first order in Day (1967) given by

$$\left.\begin{aligned} y'(x) + y(x) &= (x^2 + 2x + 1)e^{-x} \\ &+5x^2 + 8 - \int_0^x ty(t)dt, \\ &0 \le x \le 1 \\ &y(0) = 10 \end{aligned}\right\} \qquad (20)$$

The exact solution is $y(x) = 10 - xe^{-x}$

.
Let $y(x) = \sum_{j=0}^{N} a_j P_j(x)$ (21)

Where are orthogonal polynomials. Substituting (21) in to (20) and simplifying we have

$$\sum_{j=0}^{N} a_j \left(P_j^{(n)}(x) + P_j(x) + \int_0^x K(x,t) P_j(t) dt \right) = F(x) \qquad (22)$$

$$K(x,t) = t$$

and $F(x)=(x^2+2x+1)e^{-x}+5x^2+8$

Substituting (19) in (22) together with the initial condition we have (22) in matrix form of (18)

Where

$$\left. \begin{array}{c} \sigma_j(x_k) = P_j^{(n)}(x_k) + P_j(x) \\ + \int_0^x K(x_k, t) P_j(t) dt \\ k = 1,2,3,\ldots,N, \qquad j = 0,1,2,\ldots,N \end{array} \right\} \qquad (23)$$

Solving for a'^{s}_{j}

for the three orthogonal polynomials we have the solutions presented in table 1.

Example 2

Let us consider the Volterra integro-differential equation of the second order in Wazwaz (2011) given by

$$\left. \begin{array}{c} y''(x) = (1+x) + \int_0^x (x-t) y(t) dt, \\ 0 \leq x \leq 1 \\ y(0) = 1 \\ y'(0) = 1 \end{array} \right\} \qquad (24)$$

The exact solution is $y(x)=e^x$

In the similar way, Let

$$y(x) = \sum_{j=0}^{N} b_j P_j(x) \qquad (25)$$

Where are orthogonal polynomials. Substituting (25) in to (24) and simplifying we have

$$\sum_{j=0}^{N} b_j \left(P_j^{(n)}(x) - \int_0^x K(x,t) P_j(t) dt \right) = F(x) \qquad (26)$$

where $K(x,t)=(x-t)$, $n=2$ and $F(x)=(1+x)$

Equation (26) is resolved in the form of (18) using the initial conditions and the collocation points to obtain b_j

The solution is presented in Table 2 below

Table 1 Results of the Solution to Example 1 with Legendre, Laguerre and Chebyshev Polynomials and Absolute Errors

x	Exact Solution	Solution with Legendre Polynomials	Error with Legendre Polynomials	Solution with Laguerre Polynomials	Error with Laguerre Polynomials	Solution with Chebyshev Polynomials	Error with Chebyshev Polynomials
0.00	10	10	0	10	0	10	0
0.1	9.909516258	9.909516236	2.2×10^{-8}	9.909516253	5.0×10^{-9}	9.909516236	2.2×10^{-8}
0.2	9.836253849	9.836253831	1.8×10^{-8}	9.836253836	1.3×10^{-8}	9.83625383	1.9×10^{-8}
0.3	9.777754534	9.777754518	1.6×10^{-8}	9.777754529	5.0×10^{-9}	9.777754516	1.8×10^{-8}
0.4	9.731871982	9.731871966	1.6×10^{-8}	9.731871972	1.0×10^{-8}	9.731871966	1.6×10^{-8}
0.5	9.69673467	9.696734657	1.3×10^{-8}	9.696734665	5.0×10^{-9}	9.696734656	1.4×10^{-8}
0.6	9.670713018	9.670713004	1.4×10^{-8}	9.670713018	0	9.670713005	1.3×10^{-8}
0.7	9.652390287	9.652390277	1.0×10^{-8}	9.652390291	4.0×10^{-9}	9.652390277	1.0×10^{-8}
0.8	9.640536829	9.64053682	9.0×10^{-9}	9.640536824	5.0×10^{-9}	9.640536819	1.0×10^{-8}
0.9	9.634087306	9.634087297	9.0×10^{-9}	9.634087307	1.0×10^{-9}	9.634087295	1.1×10^{-8}
1.0	9.632120559	9.632120556	3.0×10^{-9}	9.63212057	1.1×10^{-8}	9.632120545	1.4×10^{-8}

Table 2 Results of the Solution to Example 2 with Legendre, Laguerre and Chebyshev Polynomials and Absolute Errors

x	Exact Solution	Solution with Legendre Polynomials	Error with Legendre Polynomials	Solution with Laguerre Polynomials	Error with Laguerre Polynomials	Solution with Chebyshev Polynomials	Error with Chebyshev Polynomials
0.00	1	0.9999999999	1×10^{-10}	0.999999996	4.0×10^{-9}	1.0	0
0.1	1.105170918	1.105170918	0	1.105170886	3.2×10^{-8}	1.105170918	0
0.2	1.221402758	1.221402759	1.0×10^{-9}	1.221402688	7.0×10^{-8}	1.221402759	1×10^{-9}
0.3	1.349858808	1.349858807	1.0×10^{-9}	1.349858705	1.03×10^{-7}	1.349858808	0
0.4	1.491824698	1.491824698	0	1.491824558	1.4×10^{-7}	1.491824699	1×10^{-9}
0.5	1.648721271	1.648721269	2.0×10^{-9}	1.648721089	1.82×10^{-7}	1.648721270	1×10^{-9}
0.6	1.8221188	1.822118801	1×10^{-9}	1.822118574	2.26×10^{-7}	1.822118800	0
0.7	2.013752707	2.013752708	1×10^{-9}	2.013752435	2.72×10^{-7}	2.013752708	1×10^{-9}
0.8	2.225540928	2.225540928	0	2.225540608	3.2×10^{-7}	2.225540929	1×10^{-9}
0.9	2.459603111	2.45960311	1×10^{-9}	2.459602742	3.69×10^{-7}	2.459603111	0
1.0	2.718281828	2.718281828	0	2.718281408	4.2×10^{-7}	2.718281829	1×10^{-9}

DISCUSSION

In this paper, we discuss the results of the collocation methods by different orthogonal polynomials for the solutions of Volterra integrodifferential equations in the general form. Like the previous studies about the collocation method of Volterra integrodifferential equations by Olumuyiwa *et al*. (2017), the standard collocation method (SCM) is used to determine the desired collocation points within an interval say, $[\vartheta, \sigma]$, to solve sample problems ranging from linear to nonlinear first and second order Volterra integro-differential (VIDEs) equations using our methods. we have employed the use of different orthogonal polynomials as test functions to obtain the solution of first and second order Volterra integrodifferential equations. MAPLE computer software have been very useful in this regard. For this purpose, the method which is used here is based on Legendre polynomials, Laguerre polynomials and Chebyshev polynomials, in order to find the approximate solutions and compare with the analytic solutions of Volterra integro-differential equations. Thus, the approximate solutions are obtained as a convergent.

polynomial series. A considerable advantage of this method is that the coefficients of the basis polynomials of the solution are obtained easily by using the computer program as mentioned above.

Example 1 is the Volterra integro-differential equation of the first order curled from Day (1967) and the exact solution is given as $y(x) = 10\text{-}xe^{-x}$ In Table1, the method constructed is used and the results obtained with Legendre polynomial, Laguerre polynomials and Chebyshev polynomials as basis functions are compared with the exact solution. The minimum absolute error is zero for all the polynomials and the maximum absolute error is 2.2×10^{-8} for both Legendre and Chebyshev polynomials while the maximun absolute error for Laguerre is 1.3×10^{-8} as compared with the exact solution. Hence, the method with Laguerre polynomials gives a better approximation as seen in Table 1.

In Table 2, Example 2 is a second order Volterra integro-differential equation curled from Wazwaz (2011) with the exact solution given as $y(x) = e^{x}$. We test the method on a second order Volterra Integro-Differential Equation and the compared values of the results of the proposed method with exact solution shows that using the Legendre polynomials as the basis functions shows the absolute error of zero to be minimum and 2.0×10^{-9} as maximum, using Laguerre polynomial as basis function, the minimum absolute error is 4.0×10^{-9} while the maximum is 4.2×10^{-7} and with Chebyshev polynomial, the solution compared with the exact solution gives a minimum absolute error of zero and the maximum absolute error is 1.0×10^{-9}

For the second order Volterra integro-differential equation too, the Legendre polynomial gives a better approximation as compared to the Chebyshev and Laguerre Polynomials.

CONCLUSION

The Laguerre polynomial offer more accurate solution for a first order Volterra integro-differential equation and the convergence of the method is impressive for all the three basis functions, which agrees with the work of Olumuyiwa *et al*. (2017).

We also observed that the solutions of the problem by using the orthogonal polynomials shows that Legendre has the best advantage with respect to the best approximate solution for a second order Volterra-Integro-differential equation, its absolute error is the smallest as compared to the rest of the orthogonal polynomials considered in this analysis. However, all the three polynomials offer good approximations.

Recommendation

We restricted the application of our constructed method to solving linear Volterra integro-differential equations of the first and second order only, however we recommend the application to non-linear higher order Volterra integro-differential equations.

We also recommend the application to Fredholm and Fredholm- Volterra integro-differential equations.

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